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Darboux Coordinates for the Hamiltonian of First Order Einstein-Cartan Gravity

N. Kiriushcheva · S.V. Kuzmin

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Abstract Based on our preliminary analysis of the Hamiltonian formulation of the first order Einstein-Cartan action (arXiv:0902.0856 [gr-qc] and arXiv:0907.1553 [gr-qc]) we derive the Darboux coordinates, which are a unique and uniform change of variables that preserve equivalence with the original action in all spacetime dimensions higher than two. Considerable simplification of the Hamiltonian formulation using the Darboux coordinates, as compared with the direct analysis, is explicitly demonstrated. Even an incomplete Hamiltonian analysis in combination with known symmetries of the Einstein-Cartan action and the equivalence of Hamiltonian and Lagrangian formulations allows us to unambiguously conclude that the *unique gauge* invariances generated by the first class constraints are *translation and rotation in the tangent space*. Diffeomorphism invariance, though a manifest invariance of the action, is not generated by the first class constraints of the theory.

Keywords Einstein-Cartan gravity · Hamiltonian · Poincaré gauge theory

1 Introduction

In this paper we continue our search for the gauge invariance of the Einstein-Cartan (EC) action using the Hamiltonian formulation of its first order form, which is valid in all spacetime dimensions (D) higher than two ($D > 2$). This investigation was started in [1–3]. The complete Hamiltonian analysis of the EC action when $D = 3$, including the restoration of gauge invariance, was performed in [1] because of the simplification of the calculations which appears in $D = 3$. In dimensions $D > 3$ the calculations are much more involved and were not completed, although the preliminary results were reported in [2]. The main goal of the present paper is the derivation of the Darboux coordinates which, as we will show,

N. Kiriushcheva (✉) · S.V. Kuzmin
Faculty of Arts and Social Science, Huron University College and Department of Applied Mathematics, University of Western Ontario, London, Canada
e-mail: nkiriush@uwo.ca

S.V. Kuzmin
e-mail: skuzmin@uwo.ca

drastically simplify the calculations in the first steps of the Dirac procedure [4–6]. We hope that additional simplifications will also occur in the later steps and that it will be possible to complete the Hamiltonian analysis for $D > 3$ and give the unique answer to the question: what is the *gauge* symmetry generated by the first class constraints.

The Hamiltonian formulation of the EC action is an old and apparently solved problem. It is claimed in many articles and included in monographs that the canonical formulation of the EC theory has already been completed and its *gauge* symmetries are Lorentz invariance and diffeomorphism (very often only the so-called "spatial" diffeomorphism). The reasons for reconsidering this claim are the following.

(A) The first order form of the EC action is a uniform formulation valid in all $D > 2$ dimensions and it seems to us very suspicious that such a drastic change of gauge invariance in different dimensions is possible, e.g. from Poincaré in the three-dimensional case (both gauge parameters with internal indices) [1, 7] to Lorentz (internal) plus diffeomorphism (external) when $D = 4$, as is stated in many papers on Hamiltonian formulations of tetrad gravity and especially in papers and monographs on Loop Quantum Gravity (LQG) [8, 9], where the spatial diffeomorphism constraint is always present irrespective of what variables are used. But it is clear even from the first steps of the Dirac procedure, as shown in [1, 2], that diffeomorphism (neither spatial diffeomorphism, as in LQG, nor full spacetime diffeomorphism) *cannot be a gauge symmetry* generated by the first class constraints of the EC action, not only in $D = 3$, but in any dimension. The claim that the spatial diffeomorphism is a *gauge* symmetry of tetrad gravity is the result of a non-canonical change of variables (see Sect. V of [10]) that was "justified" only by such "arguments" as "convenience" and a desire to accommodate the "expected" results. Of course, diffeomorphism is an invariance of the EC action (as it is manifestly generally covariant), so are the rotation and translation in internal space [11]; in fact, many other invariances can be found in the Lagrangian formalism by constructing differential identities considering, for example, linear combinations of basic differential identities (see [3]). The statement that the EC action is invariant under internal translation when $D = 3$ and is not invariant in dimensions $D > 3$ is simply wrong as this contradicts known results [11, 12]. The change of *gauge* symmetry in the Hamiltonian formulation from internal translation (the gauge parameter has an internal index) to diffeomorphism (the gauge parameter is the "world" vector) does not seem to be feasible as the first order EC action is formulated uniformly for all dimensions ($D > 2$).

Diffeomorphism is one of the invariances of the EC action; but it is not a gauge symmetry generated by the first class constraints [1, 2]. The *gauge symmetry* is a characteristic of a theory and in a Hamiltonian formalism it must be *uniquely* derived using the Dirac procedure. According to Dirac's conjecture [4] all the first class constraints of the Hamiltonian formulation are responsible for the *gauge* invariance and any gauge symmetry must be derivable from first class constraints [13] using, for example, the Castellani algorithm [14]. Only after that is it possible to answer the question posed by Matschull [15] for $D = 3$ (but which is equally well relevant in all dimensions): "what is a gauge symmetry and what is not". We are not aware of such a derivation for the Hamiltonian formulation of the EC action, similar to what was done for the Einstein-Hilbert, metric, action [16, 17], where full spacetime diffeomorphism is indeed the gauge symmetry. Some arguments that so-called spatial diffeomorphism is a gauge symmetry have been made; but this is not even a symmetry of the EC action, which is invariant under the full spacetime diffeomorphism, but not only under its spatial part separately.

(B) It was shown some time ago that the EC action is invariant under "translations and rotations in the tangent spaces" [11] with the algebra of generators that has "a more general group structure than the original Poincaré group" [12]. It differs from the original Poincaré group for $D > 3$ only by having a non-zero commutation relation between two translational generators [11, 12]. Moreover, the explicit form of the transformations of fields were given with two parameters that correspond to internal translation and rotation (see e.g. [12, 18]). These results contradict the statement that the EC Lagrangian is not invariant under translation. Recently (without referring to the Hamiltonian formalism) translational and rotational invariances of the first order EC action were derived in [3] using the iterative procedure that allows one to construct the simplest differential identities from the Euler derivatives which follow from the EC Lagrangian. This procedure is based on the Noether second theorem [19] that allows us to relate differential identities to the corresponding transformations of fields. To find a gauge invariance of the EC action, a Hamiltonian analysis is needed because it gives a unique answer to the question: what is the gauge symmetry generated by the first class constraints. However, the choice of what to call a gauge invariance is often based on different arguments, i.e. according to [20] "it is partly possible (and physically more plausible) to unify the two local gauge groups—Poincaré on the frames and general covariance". Is it possible to have the *gauge* group that unifies Poincaré and diffeomorphism? The Lagrangian and Hamiltonian formalisms are equivalent; and in the Hamiltonian formalism, the *gauge* invariance must be derived from first class constraints [13], and not be imposed from the outset. The number of constraints not only fixes the number of gauge parameters (which equals to the number of primary first class constraints) and their tensorial character, but it also defines the number of degrees of freedom (found from the number of all constraints) [21]. Thus, internal translation and diffeomorphism cannot simultaneously belong to the same *gauge* group, as the number of constraints needed to accommodate both symmetries leads to a negative number of degrees of freedom, which is physically not plausible. We need the Hamiltonian analysis to isolate a unique gauge invariance of an action, out of many other invariances, i.e., the invariance that follows from the first class constraints. We do not rely on geometrical, or physical, or any other seemingly plausible argument; and our *goal is to reveal the gauge symmetry of the EC theory using the Hamiltonian method*.

(C) The Hamiltonian analysis of the first order form of the EC action in some cases is specialized to some particular dimension, e.g. [7, 22–25]. However, such formulations might either destroy or miss some general features of the original action, which are valid in all dimensions (except the special case $D = 2$). For example, when constructing Darboux coordinates for the EC action it is artificial to introduce such variables separately for each dimension (as [25]); they have to be common for all dimensions as is the original EC action.

We are looking for Darboux coordinates which are valid in all dimensions $D > 2$. This is not a purely mathematical interest to find the most general formulation; but it has a practical reason: the EC action is formulated in all dimensions, and so the correct methods have to produce meaningful results in all dimensions simultaneously. This will guarantee that nothing is missing or misinterpreted in the physically important four-dimensional case. Examples of formulations that were designed for only particular dimensions are: when $D = 3$, the treatment of the EC action was based on similarities (but not equivalence [15]) with the Chern-Simons action [7]; the construction of Darboux coordinates by Bañados and Contreras [25] works only in the $D = 4$ case and allows for neither the consideration of the $D = 3$ limit nor generalization on dimensions higher than four. To have the correct Hamiltonian formulation of the EC action and to find its unique gauge invariance in the physically interesting $D = 4$ case, we have to perform the analysis using an approach valid in all dimensions. An important property of using a formulation valid in all dimensions is the possibility to check the $D = 3$ limit at all stages of the calculations. The Hamiltonian formulation in the $D = 3$ case can be carried out without difficulty because of simplifications occurring when $D = 3$;

it gives the consistent result and a simple Lie algebra of Poisson brackets (PBs) among the first class constraints [1]. In higher dimensions we can expect that some modifications of the Poincaré algebra will appear (we argued in [2] that the only possible modification is the non-zero PB among two translational constraints, which is exactly what happens in [11, 12]); but such modifications must vanish in the $D=3$ limit. The calculation of constraints and their PB algebra in the Hamiltonian formulation of the EC action is still quite involved even after drastic simplification due to introduction of Darboux coordinates; and so the possibility of checking the consistency of the results by considering the $D=3$ limit at all stages of calculations is extremely important. Thus, *we do not specialize our analysis to a particular dimension*.

The construction of Darboux coordinates which are uniform for all dimensions and simplify the Hamiltonian analysis is the main goal of the present article.

The paper is organized as follows. In Sect. 2 we establish notation and provide arguments to support our expectation that the Darboux coordinates exist for the theory under consideration. In Sect. 3, based on the result of the direct Hamiltonian analysis [2], we derive the Darboux coordinates. In Sect. 4, we show that introduction of Darboux coordinates allows one to perform the Lagrangian or Hamiltonian reduction in a much simpler manner than the Hamiltonian reduction in the direct Hamiltonian approach of [2], and to attack the most involved calculations: finding PBs among secondary first class constraints (or equivalently, to prove the closure of the Dirac procedure) which are needed to find gauge transformations of the EC action in the Hamiltonian formalism. These calculations are briefly outlined. In particular, using the Dirac brackets, we demonstrate that there is a strong indication that in all dimensions the PB between translational and rotational constraints are the same and coincide with the corresponding part of the Poincaré algebra known for the $D=3$ case (the same conclusion was made in [2] using the Castellani algorithm [14]). In Sect. 5 the results are summarized and the conclusion about the *gauge* invariance of the EC action is made. The properties of some combinations of fields that considerably simplify the calculations are collected in Appendix A. In Appendix B the solution of the equation that arises in the course of the Lagrangian/Hamiltonian reduction is given.

2 Notation and Expectations

In [2] we considered the Hamiltonian formulation of the Einstein-Cartan action by direct application of the Dirac procedure to its first order form [26, 27]

$$I_{EC} = -\int d^D x\, e \left(e^{\mu(\alpha)} e^{\nu(\beta)} - e^{\nu(\alpha)} e^{\mu(\beta)}\right) \left(\omega_{\nu(\alpha\beta),\mu} + \omega_{\mu(\alpha\gamma)} \omega_{\nu}{}^{(\gamma}{}_{\beta)}\right), \tag{1}$$

where the covariant N-beins $e_{\gamma(\rho)}$ and the connections $\omega_{\nu(\alpha\beta)}$ ($\omega_{\nu(\alpha\beta)} = -\omega_{\nu(\beta\alpha)}$) are treated as independent fields in all spacetime dimensions ($D > 2$), and $e = \det(e_{\gamma(\rho)})$.[1] Greek letters indicate covariant indices $\alpha = 0, 1, 2, \ldots, (D-1)$. Indices in brackets $(\ldots)$ denote the internal ("Lorentz") indices, whereas indices without brackets are external or "world" indices. Internal and external indices are raised and lowered by the Minkowski tensor $\tilde{\eta}_{\alpha\beta} = (-, +, +, \ldots)$ and the metric tensor $g_{\mu\nu} = e_{\mu(\alpha)} e_{\nu}^{(\alpha)}$, respectively (we use a tilde for

[1] Usually variables $e_{\gamma(\rho)}$ and $\omega_{\nu(\alpha\beta)}$ are named tetrads and spin connections, but such names are specialized for $D=4$. As we consider the Hamiltonian formulation in any dimension ($D > 2$), we will call $e_{\gamma(\rho)}$ and $\omega_{\nu(\alpha\beta)}$ N-beins and connections, respectively.

any combination with only internal indices and do not use brackets in such cases, except to indicate antisymmetrization in pairs of indices). N-beins are invertible: $e^{\mu(\alpha)}e_{\mu(\beta)} = \tilde{\delta}^{\alpha}_{\beta}$, $e^{\mu(\alpha)}e_{\nu(\alpha)} = \delta^{\mu}_{\nu}$.

The Lagrangian density of (1), after integration by parts, can be written in the following form

$$L_{EC}\left(e_{\mu(\alpha)}, \omega_{\mu(\alpha\beta)}\right) = eB^{\gamma(\rho)\mu(\alpha)\nu(\beta)}e_{\gamma(\rho),\mu}\omega_{\nu(\alpha\beta)} - eA^{\mu(\alpha)\nu(\beta)}\omega_{\mu(\alpha\gamma)}\omega_{\nu}{}^{(\gamma}{}_{\beta)}, \tag{2}$$

where the functions $A^{\mu(\alpha)\nu(\beta)}$ and $B^{\gamma(\rho)\mu(\alpha)\nu(\beta)}$ are defined as

$$A^{\mu(\alpha)\nu(\beta)} = e^{\mu(\alpha)}e^{\nu(\beta)} - e^{\nu(\alpha)}e^{\mu(\beta)}, \qquad \frac{\delta}{\delta e_{\gamma(\rho)}}\left(eA^{\mu(\alpha)\nu(\beta)}\right) = eB^{\gamma(\rho)\mu(\alpha)\nu(\beta)}. \tag{3}$$

The properties of the functions $A^{\mu(\alpha)\nu(\beta)}$ and $B^{\gamma(\rho)\mu(\alpha)\nu(\beta)}$ and their further generations that considerably simplify the calculations are collected in Appendix A.

For the Hamiltonian formulation, where we have to separate spatial and temporal *indices* (not separating spacetime itself into space and time),[2] we use 0 for an external "time" index (and (0) for an internal "time" index) and Latin letters for "spatial" external indices $k = 1, 2, \ldots, (D-1)$ ((k) for "spatial" internal indices).

In Progress Report [2] (the references to equations from [2] are indicated as (R#)) we demonstrated that after performing the Hamiltonian reduction (i.e. eliminating part of the variables by solving the second class constraints) in all dimensions, the canonical part of the total Hamiltonian is a linear combination of secondary constraints (called "rotational" $\chi^{0(\alpha\beta)}$ and "translational" $\chi^{0(\sigma)}$ constraints, see (R152))

$$H_{reduced}\left(e_{\mu(\rho)}, \pi^{\mu(\rho)}, \omega_{0(\alpha\beta)}, \Pi^{0(\alpha\beta)}\right) = \pi^{0(\rho)}\dot{e}_{0(\rho)} + \Pi^{0(\alpha\beta)}\dot{\omega}_{0(\alpha\beta)} + H_c, \tag{4}$$

where the canonical Hamiltonian (up to a total spatial derivative) is

$$H_c = -\omega_{0(\alpha\beta)}\chi^{0(\alpha\beta)}\left(e_{\mu(\rho)}, \pi^{k(\rho)}\right) - e_{0(\sigma)}\chi^{0(\sigma)}\left(e_{\mu(\rho)}, \pi^{k(\rho)}\right). \tag{5}$$

This form of the Hamiltonian is not new and appeared for the first time in [29], but it was based on very general arguments and the explicit form of constraints was not given. For $D = 4$ the reduced Hamiltonian[3] with the same set of canonical variables (4) was obtained in [27], but the closure of the Dirac procedure was considered only after the authors switched to a different set of variables: lapse and shift functions. In later works this form of the Hamiltonian (5) can be found at most on some intermediate steps of the Hamiltonian analysis (i.e. see [24, 25]) after which the change of initial variables is performed, e.g. from $e_{\mu(\rho)}$ to $e_{k(n)}$ and lapse and shift functions [31]. Usual arguments for such changes are "it is

[2]If one writes, for example, the equations of motion of a covariant theory in components the covariance is not lost, though it is not manifest. The common statement as in [28]: "Unfortunately, the canonical treatment breaks the symmetry between space and time in general relativity and the resulting algebra of constraints is not the algebra of four diffeomorphism" is groundless. In the Hamiltonian formulation of General Relativity the covariance is not manifest, but it is not broken as the gauge symmetry of the Einstein-Hilbert action, diffeomorphism, is recovered in a manifestly covariant form for the second order [16, 17] and the first order [10] formulations.

[3]This is the result of the Hamiltonian reduction, i.e. solving second class constraints and eliminating the corresponding pairs of canonical variables. The reduced Hamiltonian obtained in such a way should not be confused with reduction based on solving first class constraints [30], an operation that contradicts the Dirac procedure.

more convenient" [32] or "it is useful" [27] but canonicity was never discussed. We showed in [10] that the change of variables from $e_{\mu(\rho)}$ to $e_{k(n)}$ and lapse and shift functions is not canonical.

All the PBs among primary ($\pi^{0(\rho)}$, $\Pi^{0(\alpha\beta)}$) and among primary and secondary constraints ($\chi^{0(\alpha\beta)}$, $\chi^{0(\sigma)}$) are zero; and the PB between two rotational constraints in all dimensions is

$$\left\{\chi^{0(\alpha\beta)}, \chi^{0(\mu\nu)}\right\}_{D>2} = \frac{1}{2}\tilde{\eta}^{\beta\mu}\chi^{0(\alpha\nu)} - \frac{1}{2}\tilde{\eta}^{\alpha\mu}\chi^{0(\beta\nu)} + \frac{1}{2}\tilde{\eta}^{\beta\nu}\chi^{0(\mu\alpha)} - \frac{1}{2}\tilde{\eta}^{\alpha\nu}\chi^{0(\mu\beta)}, \tag{6}$$

which corresponds to Lorentz rotation in the tangent space. It is necessary to find the remaining PBs:

$$\left\{\chi^{0(\alpha\beta)}, \chi^{0(\rho)}\right\}_{D>3} = ?, \tag{7}$$

$$\left\{\chi^{0(\rho)}, \chi^{0(\gamma)}\right\}_{D>3} = ?. \tag{8}$$

In the $D = 3$ case, the calculation of (7)–(8) is simple [1] and leads to:

$$\left\{\chi^{0(\alpha\beta)}, \chi^{0(\rho)}\right\}_{D=3} = \frac{1}{2}\tilde{\eta}^{\beta\rho}\chi^{0(\alpha)} - \frac{1}{2}\tilde{\eta}^{\alpha\rho}\chi^{0(\beta)}, \tag{9}$$

$$\left\{\chi^{0(\rho)}, \chi^{0(\gamma)}\right\}_{D=3} = 0, \tag{10}$$

i.e. the PB algebra of secondary constraints (6), (9) and (10) is a true Poincaré algebra; and a complete set of first class constraints when using the Castellani algorithm leads to the rotational and translational invariance in the tangent space [1]. In higher dimensions, even knowledge only of the primary constraints is enough to conclude that it is impossible to have diffeomorphism invariance following from the first class constraints and the *gauge* parameters must possess internal indices, i.e. they lead to rotation and translation in the internal space. Whether it is true or modified Poincaré algebra, can only be found after the PBs (7)–(8) are calculated.

In higher dimensions, calculation of the PBs of (7)–(8) is very laborious because of the complexity of the constraints [2]. Nevertheless, these PBs are needed to prove closure of the Dirac procedure, and to find the transformations that are produced by the first class constraints, i.e. to answer the question (in the Hamiltonian formalism): which symmetry (from an infinite set of symmetries of the EC action [3]) is the *gauge* symmetry of the EC action.

In the conclusion of [2] we discussed possible modifications of the algebra of the PBs in dimensions $D > 3$, based on the assumption that despite a more complicated form of constraints, the algebra of secondary constraints remains Poincaré, as in the $D = 3$ case, or becomes the modified Poincaré algebra. We showed that in such cases, the Lagrangian corresponding to the reduced Hamiltonian (5) remains invariant under the same transformations. This Lagrangian can be obtained by performing the inverse Legendre transformations and it gives us just a different first order formulation (with respect to temporal derivatives, not to spatial) of the original EC theory (see (R162))

$$L_{reduced}\left(e_{\mu(\alpha)}, \pi^{k(\rho)}, \omega_{0(\alpha\beta)}\right) = \pi^{k(\rho)}\dot{e}_{k(\rho)} + \omega_{0(\alpha\beta)}\chi^{0(\alpha\beta)}\left(e_{\mu(\rho)}, \pi^{k(\rho)}\right) + e_{0(\sigma)}\chi^{0(\sigma)}\left(e_{\mu(\rho)}, \pi^{k(\rho)}\right). \tag{11}$$

This form of the Lagrangian can also be seen as one which leads directly to the Hamiltonian formulation. The idea is not new—Dirac used such modifications in metric gravity to modify the Lagrangian in order to have simple primary constraints. From the Hamiltonian analysis further simplifications are found that can be implemented also at the Lagrangian level (see [33], (4.4.12) and detailed calculations leading to this change in [17]).

In the second order metric gravity there are no second class constraints; and the modification of the Lagrangian was done by adding some total temporal or spatial derivatives. In the case of the first order EC action, the situation is more complicated because of the presence of the second class constraints, which are eliminated in the course of the Dirac procedure. Equations of motion of (11) cannot be compared with equations for (1) because we have a different number of variables. In metric gravity if we perform an inverse Legendre transformation for the Dirac Hamiltonian and eliminate momenta; we will obtain Einstein's equations written in second order form. Here, after elimination of momenta, we are left only with variables $e_{\mu(\alpha)}$ and $\omega_{0(\alpha\beta)}$. So, we either have to eliminate $\omega_{0(\alpha\beta)}$ to compare with equations in the second order formulation or solve (1) for the spatial components of connections $\omega_{k(\alpha\beta)}$.[4] These calculations are very involved compared with the elimination of covariant connections; but such calculations are a good consistency check of the Hamiltonian. In the Lagrangian formalism, $\pi^{k(\rho)}$ as well as $\omega_{0(\alpha\beta)}$ are just auxiliary variables that can be eliminated using their equations of motion (exactly as $\omega_{\mu(\alpha\beta)}$ can be eliminated in (1)), which leads back to the second order EC action. The Lagrangian (11) gives the first order formulation of the EC action that differs in field content from (1), but they both are equivalent to the second order form of the EC action (after elimination of auxiliary fields). However, the first order form (11) leads directly to the Hamiltonian (4).

We can say that the following operations were performed:

$$L_{EC} \to H \underset{\text{Hamiltonian/Dirac reduction}}{\Longrightarrow} H_{reduced} \text{ (see (4))} \to L_{reduced} \text{ (see (11))}. \tag{12}$$

This suggests (because the Hamiltonian and Lagrangian formalisms must, of course, lead to the same description of a system if the reductions are performed correctly [35]) that it should be possible to obtain such a reduced Lagrangian (11) directly from the EC action that can simplify the calculations of $H_{reduced}$ compared to the direct calculations [2] performed for (2). Perhaps it can simplify the calculations of the remaining PBs among the secondary constraints (7)–(8) and the corresponding gauge transformations. Such a modification of the Lagrangian was discussed by Faddeev and Jackiw in [36]. The existence of such transformations follows from the Darboux theorem and this is what is called Darboux transformations or Darboux coordinates; this approach is also known under name "symplectic". After the paper [36] appeared, the symplectic approach and methods of finding such transformations attracted considerable interest (see, e.g. [37–39] for more details and simple examples). Many results that were obtained before by other methods were reconsidered using this approach (even though sometimes it leads to more complicated calculations than the original formulations); but in the case we are considering here (the first order EC action), it leads to simplifications. The equivalence of this approach with the Dirac procedure was investigated. For some models "non-equivalence" of the Dirac and symplectic methods was found (see, e.g. [37, 38], where the role of second class constraints was emphasized, and [39], where the observation of non-equivalence leads the authors to the conclusion about deficiency of the Dirac procedure; some doubts about the symplectic method were expressed

[4] The Lagrangian obtained in such a way can be called "one and half" order and its Hamiltonian formulation is also free from second class constraints but calculations with such a Lagrangian and even elimination of spatial components of connections is not a simple task and does not give advantages compared with the direct calculations described in [2]. The term "one and half" formulation exists in supergravity [34] but it has a different meaning.

in [38]). All examples of non-equivalence are related to the systems with second class constraints, which is exactly the case of the EC action. Here we follow the advice of Faddeev and Jackiw [36]: "If at some stage the elimination is too difficult to carry out, one may resort to Dirac's approach".

We use our preliminary Hamiltonian analysis [1, 2] to find the Darboux transformations. The term of our interest, "kinetic" part of the Lagrangian, is

$$eB^{m(\rho)0(\alpha)k(\beta)}\omega_{k(\alpha\beta)}e_{m(\rho),0}$$

(based on antisymmetry properties of B: if one external index is temporal, the only non-zero contributions are possible if the rest of the external indices are spatial).

We want to find such variables that diagonalize the "kinetic part", and the rest of the variables can be eliminated. In other words, we are looking for Darboux coordinates such that

$$L_{EC} \underset{\text{Darboux coordinates}}{\Longrightarrow} L_{EC(D)} \underset{\text{Lagrangian reduction}}{\Longrightarrow} L_{reduced}$$

with

$$L_{reduced} = L_{reduced} \text{ (see (11))} \rightarrow H_{reduced} \text{ (see (4))}. \tag{13}$$

In the next section, based on the result of the direct Hamiltonian analysis [2], we derive the following Darboux transformations for spatial connections (the temporal connections $\omega_{0(\alpha\beta)}$, as well as the basic variables $e_{\gamma(\rho)}$, N-beins, remain unaltered)

$$\omega_{m(\alpha\beta)} = N_{m(\alpha\beta)0n(\sigma)}F^{n(\sigma)} + e_{p(\alpha)}e_{q(\beta)}\hat{\Sigma}_m{}^{(pq)} = \omega_{m(\alpha\beta)}(F) + \omega_{m(\alpha\beta)}(\hat{\Sigma}), \tag{14}$$

where $N_{m(\alpha\beta)0n(\sigma)}$ is a non-linear, algebraic (without derivatives) combination of N-beins, which is antisymmetric in $\alpha\beta$ whose explicit form is given by (43). The field $\hat{\Sigma}_m{}^{(pq)}$ is antisymmetric ($\hat{\Sigma}_m{}^{(pq)} = -\hat{\Sigma}_m{}^{(qp)}$) and traceless ($\hat{\Sigma}_m{}^{(mq)} = \hat{\Sigma}_m{}^{(pm)} = 0$) with all indices being external ("world") and spatial. (Here and below we will use "hat" for combinations with only external indices; and brackets are used to indicate antisymmetrization in pairs of external indices.) The transformation (14) is invertible, valid in all dimensions ($D > 2$) and preserves the $D = 3$ limit [1]. Note that the number of components of $\hat{\Sigma}_m{}^{(pq)}$ plus $F^{n(\sigma)}$ is the same as for $\omega_{m(\alpha\beta)}$ in all dimensions (the number of independent components of a field, using the notation of [33], is indicated by square brackets [*field*]):

$$\left[\omega_{m(\alpha\beta)}\right] = \frac{1}{2}D\left(D-1\right)\left(D-1\right), \tag{15}$$

$$\left[F^{n(\sigma)}\right] = D\left(D-1\right), \tag{16}$$

$$\left[\hat{\Sigma}_m{}^{(pq)}\right] = \frac{1}{2}D\left(D-1\right)\left(D-3\right)' \tag{17}$$

which gives

$$\left[\omega_{m(\alpha\beta)}\right] = \left[F^{n(\sigma)}\right] + \left[\hat{\Sigma}_m{}^{(pq)}\right]. \tag{18}$$

In the discussion of Darboux coordinates specialized to the $D = 4$ case appearing in [25], a different field is introduced, $\hat{\lambda}_{km} = \hat{\lambda}_{mk}$, instead of our $\hat{\Sigma}_m{}^{(pq)}$. The number of components is $[\hat{\lambda}_{km}] = \frac{1}{2}D(D-1)$ which gives the correct balance of fields (see (18)) only in the $D=4$ case (as in this dimension $[\hat{\lambda}_{km}] = [\hat{\Sigma}_m{}^{(pq)}] = 6$), but supports neither a $D=3$ limit nor a generalization to higher dimensions. The uniform description of the EC action in all dimensions is broken by such variables.

In Sect. 4, we show that the transformation (14) not only diagonalizes the "kinetic" part of the Lagrangian (terms with temporal derivatives of N-beins), as a consequence of the following properties

$$eB^{k(\rho)0(\alpha)m(\beta)}\omega_{m(\alpha\beta)}(F) = F^{k(\rho)}, \qquad B^{k(\rho)0(\alpha)m(\beta)}\omega_{m(\alpha\beta)}(\hat{\Sigma}) = 0; \tag{19}$$

but it also provides a separation of variables that allows one to perform the Lagrangian or Hamiltonian reduction in a much simpler manner than in the direct Hamiltonian approach of [2] (i.e. to eliminate the field $\hat{\Sigma}_m{}^{(pq)}$, which corresponds to solution of the secondary second class constraints in the Hamiltonian analysis). After elimination of $\hat{\Sigma}_m{}^{(pq)}$, the Hamiltonian (4)–(5), where only the first class constraints are present, can be simply read off from the reduced first order Lagrangian (11). The Hamiltonian and the first class constraints that were obtained after long and cumbersome calculations in the direct approach of [2], which started from (2), can be found almost immediately when Darboux coordinates are introduced. We show that simplifications due to the introduction of the Darboux coordinates allows us to attack the most involved calculations in the direct approach: finding the remaining PBs (7)–(8) among secondary constraints (or equivalently, to prove the closure of the Dirac procedure), which is needed to find the gauge transformations of the EC action in the Hamiltonian formalism. These calculations will be briefly outlined. In particular, we demonstrate that there is a strong indication that in all dimensions the PB between translational and rotational constraints are the same and coincide with the corresponding part of the Poincaré algebra (9) known for the $D=3$ case [1] (the same conclusion was made in [2] using arguments based on the Castellani algorithm):

$$\left\{\chi^{0(\alpha\beta)}, \chi^{0(\rho)}\right\}_{D>2} = \frac{1}{2}\tilde{\eta}^{\beta\rho}\chi^{0(\alpha)} - \frac{1}{2}\tilde{\eta}^{\alpha\rho}\chi^{0(\beta)}. \tag{20}$$

3 Derivation of Darboux Coordinates Using a Preliminary Hamiltonian Analysis of the Einstein-Cartan Action

Direct application of the Dirac procedure to the first order formulation of the Einstein-Cartan action without specialization to a particular dimension was discussed in [2] where after performing the Hamiltonian reduction (that is, elimination of second class constraints) the total Hamiltonian (4)–(5) was obtained. However, these calculations are extremely laborious and on the last stage (closure of the Dirac procedure) become almost unmanageable with the exception of the $D=3$ case [1]. But this preliminary Hamiltonian analysis is indispensable because it allows us to find variables at the Lagrangian level that drastically simplify the first steps of the calculations. At the Lagrangian level, only the invertability of the change of variables is usually checked; but it might happen that the change of variables, even being invertible, is not canonical in the Hamiltonian formulation and equivalence with the original Lagrangian is lost. That is why, in finding new variables, it is important to rely on the Hamiltonian analysis, especially when working with systems which have first and second class constraints. We perform a classification of fields according to their relation to the constraints arising in the Hamiltonian formulation. This specific role of different fields can also be used at the Lagrangian level and, in particular, allows one to find Darboux coordinates that preserve equivalence with the original action and are helpful in reducing the amount of calculation to be performed. We will use the classification of fields that corresponds to the classification of constraints. *Primary* constraints (using a notion introduced by Anderson and Bergmann [40]), especially *first class* (a notion introduced by Dirac [4]), play the most important role in the Hamiltonian formulation and define the tensorial character of the gauge parameters (see Sect. V of [10]); so we call such variables "*primary variables*" if in the Hamiltonian formulation the corresponding momenta enter the primary first class constraints. For the first order Einstein-Cartan action the primary variables are $e_{0(\alpha)}$ and $\omega_{0(\alpha\beta)}$ [2]. *Second class* constraints (using Dirac's classification), irrespective of their generation (primary, secondary, etc.), can be solved for pairs of phase-space variables (this is the Hamiltonian reduction). We call such variables "*second class variables*", i.e. variables that can be eliminated at the Hamiltonian level. In the case of the first order Einstein-Cartan action, the second class variables are $\omega_{k(\alpha\beta)}$ [2]. Primary and second class variables of the first order EC action could already be identified from the results of the first gauge-free (without *a priori* choice of a particular gauge) Hamiltonian formulation for $D=4$ [27]. The importance of careful preliminary analysis before doing the Lagrange reduction was emphasized in [41]: "it seems important to develop reduction procedure within Lagrangian formulation—in a sense similar to the Dirac procedure in the Hamiltonian formulation—that may allow one to reveal the hidden structure of the Euler-Lagrange equations of motion in a constructive manner".

The above classification is crucial because the equivalence of the Lagrangian and Hamiltonian methods dictates that if for gauge theories changes involving primary variables are very restrictive at the Hamiltonian level [10], then the same must be true also at the Lagrangian level. An arbitrary change of variables can lead to the loss of equivalence of two formulations even for changes which are invertible, that is the sufficient condition only for nonsingular systems. The second class variables can be eliminated and there is more freedom to redefine them; but this redefinition has to be such that their elimination does not modify the PBs for the remaining fields if this is what happens in the Hamiltonian formalism. This imposes some restrictions; and even in this case, the invertability of transformations is only a necessary condition. In particular, in the construction of the Darboux coordinates for second class variables (which can be a complicated expression) they have to be independent of the primary variables, i.e. their variation with respect to primary variables must be zero in order to preserve the original independence of primary and second class variables. From these arguments it is clear that the Hamiltonian analysis is indispensable if one wants to modify the Lagrangian while keeping its equivalence with the original one (for the gauge-invariant systems). For example, the original independence of primary and second class variables, $\frac{\delta\omega_{k(\alpha\beta)}}{\delta e_{0(\rho)}} = \frac{\delta\omega_{k(\alpha\beta)}}{\delta\omega_{0(\rho\sigma)}} = 0$, should be preserved even after a change of variables related the Darboux coordinates to $\omega_{k(\alpha\beta)}$ is introduced.

In this section, we describe the construction of Darboux coordinates for the EC action and also illustrate the general points mentioned above. Our goal is to find the Darboux coordinates for the second class fields that simplify the Hamiltonian analysis. In [2], in the course of the Hamiltonian reduction, all $\omega_{k(\alpha\beta)}$ were eliminated by solving the second class constraints: one part by solving the primary constraints and another part that involved primary and secondary constraints. So, it would be preferable to find such a representation of $\omega_{k(\alpha\beta)}$ that separates its components into exactly two classes of variables, as in the Hamiltonian they were mixed leading up to quite long calculations. We want to decouple them, i.e. we have to find such a transformation of an "auxiliary", second class, field $\omega_{k(\alpha\beta)}$ (Darboux coordinates) that diagonalizes the "kinetic part" and separate variables that can be eliminated by a Lagrangian reduction. Note that such a separation automatically appears in the $D=3$ case [1].

The direct Hamiltonian analysis of the first order EC action (1) starts by introducing momenta conjugate to all independent variables. In (2) the only term that has "velocities" is

$$L\left(e_{\gamma(\rho),0}\right) = eB^{\gamma(\rho)0(\alpha)\nu(\beta)}e_{\gamma(\rho),0}\omega_{\nu(\alpha\beta)} \tag{21}$$

and so the momenta corresponding to $e_{\gamma(\rho)}$ are defined as

$$\pi^{\gamma(\rho)} = \frac{\delta L}{\delta e_{\gamma(\rho),0}} = eB^{\gamma(\rho)0(\alpha)\nu(\beta)}\omega_{\nu(\alpha\beta)}. \tag{22}$$

The only non-zero contributions, based on antisymmetry properties of $B^{\gamma(\rho)\mu(\alpha)\nu(\beta)}$ (see Appendix A and also [2, 3]), are

$$\pi^{k(\rho)} = eB^{k(\rho)0(\alpha)m(\beta)}\omega_{m(\alpha\beta)} \tag{23}$$

or upon separating $\pi^{k(\rho)}$ and $\omega_{m(\alpha\beta)}$ into "space" and "time" components

$$\pi^{k(n)} = eB^{k(n)0(p)m(q)}\omega_{m(pq)} + 2eB^{k(n)0(q)m(0)}\omega_{m(q0)}\ , \tag{24}$$

$$\pi^{k(0)} = eB^{k(0)0(p)m(q)}\omega_{m(pq)}. \tag{25}$$

Equations (24) and (25) lead to two primary second class constraints

$$\phi^{k(n)} = \pi^{k(n)} - eB^{k(n)0(p)m(q)}\omega_{m(pq)} - 2eB^{k(n)0(q)m(0)}\omega_{m(q0)} \approx 0, \tag{26}$$

$$\phi^{k(0)} = \pi^{k(0)} - eB^{k(0)0(p)m(q)}\omega_{m(pq)} \approx 0. \tag{27}$$

After introducing the following notation (see [2])

$$\gamma^{k(n)} \equiv e^{k(n)} - \frac{e^{k(0)}e^{0(n)}}{e^{0(0)}}, \qquad \gamma^{k(n)}e_{p(n)} = \delta^k_p, \qquad \gamma^{k(n)}e_{k(m)} = \tilde{\delta}^n_m,$$

$$E^{k(p)m(q)} \equiv \gamma^{k(p)}\gamma^{m(q)} - \gamma^{k(q)}\gamma^{m(p)}, \qquad I_{m(q)n(r)} \equiv \frac{1}{D-2}e_{m(q)}e_{n(r)} - e_{m(r)}e_{n(q)},$$

$$E^{k(p)m(q)}I_{m(q)n(r)} = \delta^k_n\tilde{\delta}^p_r,$$

equation (26) (because it is a second class constraint in the Hamiltonian analysis) can be solved for $\omega_{k(q0)}$ (see (R46))

$$\omega_{k(q0)} = -\frac{1}{2ee^{0(0)}}I_{k(q)m(p)}\pi^{m(p)} - \frac{e^{0(p)}}{2e^{0(0)}}I_{k(q)m(p)}E^{m(a)n(b)}\omega_{n(ab)} + \frac{e^{0(a)}}{e^{0(0)}}\omega_{k(aq)} \tag{28}$$

and (27) can be written in the following form

$$\pi^{k(0)} = -ee^{0(0)}E^{k(p)m(q)}\omega_{m(pq)}. \tag{29}$$

When $D = 3$ (and only when $D = 3$) (29) can be solved for $\omega_{m(pq)}$ and in equation (26) the terms proportional to the connections $\omega_{m(pq)}$ (with all "space" indices) cancel out, leading to the separation of these two equations, (26) and (27), into equations containing only $\omega_{m(pq)}$ and $\omega_{k(p0)}$, respectively. In addition, when $D = 3$, some terms in the Lagrangian disappear (see [1]). That is why the Hamiltonian analysis for $D = 3$ becomes so simple, and so is the derivation of the gauge transformations. In particular, when $D=3$, (29) can be solved because $[\pi^{k(0)}]=[\omega_{m(pq)}]=2$ (see [1]). Equation (29) for $D=3$ is represented by two equations for two independent components of $\omega_{m(pq)}$: $\omega_{1(12)}$ and $\omega_{2(12)}$. It can be solved for $\omega_{1(12)}$ and $\omega_{2(12)}$; and the solution can be written in "covariant" form

$$\omega_{k(qp)} = \frac{1}{2ee^{0(0)}} I_{k(q)m(p)}\pi^{m(0)}. \tag{30}$$

In higher dimensions, (29) cannot be solved in the same way. We showed in [2] that it is necessary to consider it together with the secondary constraints. In solving these constraints the combination of the form $\gamma^{m(n)}\omega_{m(pq)} = \tilde{\omega}^n{}_{(pq)}$ was very useful (in $\tilde{\omega}^n{}_{(pq)}$ all indices are internal), because the "trace" of this combination[5] is proportional to $\pi^{k(0)}$

$$\pi^{k(0)} = 2ee^{0(0)}\gamma^{k(p)}\tilde{\omega}^q{}_{(qp)}. \tag{31}$$

This suggests the introduction of variables that allow one to single out the contribution of (31), which is obviously the separation of $\tilde{\omega}_{n(pq)}$ into the trace, $\tilde{V}_q$, and the traceless, $\tilde{\Omega}_{n(pq)}$, parts

$$\tilde{\omega}_{n(pq)} = \tilde{\Omega}_{n(pq)} + \frac{1}{D-2}\left(\tilde{\eta}_{np}\tilde{V}_q - \tilde{\eta}_{nq}\tilde{V}_p\right), \tag{32}$$

or equivalently

$$\omega_{m(pq)} = e_m^{(n)}\tilde{\Omega}_{n(pq)} + \frac{1}{D-2}\left(e_{m(p)}\tilde{V}_q - e_{m(q)}\tilde{V}_p\right) \tag{33}$$

(where we have used $\tilde{\omega}_{n(pq)} = \tilde{\eta}_{nm}\tilde{\omega}^m{}_{(pq)}$).

The variable $\tilde{\Omega}_{n(pq)}$ is an antisymmetric ($\tilde{\Omega}_{n(pq)} = -\tilde{\Omega}_{n(qp)}$) and traceless ($\tilde{\Omega}^p{}_{(pq)} = \tilde{\eta}^{np}\tilde{\Omega}_{n(pq)} = 0$) field with all indices being internal. The necessary condition for any field redefinition (before checking the invertability) is that the number of fields is preserved, which is satisfied in our case because $[\tilde{\omega}_{n(pq)}] = [\tilde{\Omega}_{n(pq)}] + [\tilde{V}_q]$ in all dimensions. It is not difficult to demonstrate the invertability of (33). Contracting (32) with $\gamma^{m(p)}$ (or equally well with $\gamma^{m(q)}$) we obtain

$$\tilde{V}_p = \tilde{\omega}^q{}_{(qp)}. \tag{34}$$

Now contracting (33) with $\gamma^{m(k)}$ we find

$$\tilde{\Omega}^k{}_{(pq)} = \gamma^{m(k)}\omega_{m(pq)} - \frac{1}{D-2}\left(\tilde{\delta}^k_p\tilde{V}_q - \tilde{\delta}^k_q\tilde{V}_p\right) \tag{35}$$

and using (34)

$$\tilde{\Omega}^k{}_{(pq)} = \gamma^{m(k)}\omega_{m(pq)} - \frac{1}{D-2}\left(\tilde{\delta}^k_p\tilde{\omega}^n{}_{(nq)} - \tilde{\delta}^k_q\tilde{\omega}^n{}_{(np)}\right), \tag{36}$$

or in terms of the original connections

$$\tilde{\Omega}^k{}_{(pq)} = \gamma^{m(k)}\omega_{m(pq)} - \frac{1}{D-2}\left(\tilde{\delta}^k_p\gamma^{m(n)}\omega_{m(nq)} - \tilde{\delta}^k_q\gamma^{m(n)}\omega_{m(np)}\right). \tag{37}$$

[5] For the original connection $\omega_{m(pq)}$, antisymmetric in internal indices, such a "trace" cannot be defined. We need combinations with all indices of the same "nature" and $\tilde{\omega}^n{}_{(pq)}$ (with all indices being internal) provides such a combination, which arises naturally in the direct Hamiltonian analysis [2].

We then see that (33) is invertible. It is also easy to show that (37) is traceless.

We substitute (33) into (28) to express the connection $\omega_{k(q0)}$ in terms of new fields

$$\omega_{k(q0)} = -\frac{1}{2ee^{0(0)}} I_{k(q)m(p)} \pi^{m(p)} - \frac{e^{0(p)}}{2e^{0(0)}} I_{k(q)m(p)} E^{m(a)n(b)} \omega_{n(ab)} \left(\tilde{\Omega}, \tilde{V}\right) + \frac{e^{0(a)}}{e^{0(0)}} \omega_{k(aq)} \left(\tilde{\Omega}, \tilde{V}\right)$$

that upon substitution of $\omega_{n(ab)}(\tilde{\Omega}, \tilde{V})$ and simple contractions gives

$$\omega_{k(q0)} = -\frac{1}{2ee^{0(0)}} I_{k(q)m(p)} \pi^{m(p)} + \frac{D-3}{D-2} e_{k(0)} \tilde{V}_q + \frac{e^{0(p)}}{e^{0(0)}} e_k^{(n)} \tilde{\Omega}_{n(pq)}. \tag{38}$$

Using (31) and (34), we can express $\tilde{V}_q$ in terms of $\pi^{k(0)}$ (this is also linear in auxiliary fields redefinition with $[\pi^{k(0)}] = [\tilde{V}_p]$ in all dimensions)

$$\tilde{V}_q = \frac{1}{2ee^{0(0)}} e_{k(q)} \pi^{k(0)}. \tag{39}$$

Finally, we obtain

$$\omega_{k(q0)} = -\frac{1}{2ee^{0(0)}} I_{k(q)m(p)} \pi^{m(p)} + \frac{D-3}{D-2} e_{k(0)} \frac{1}{2ee^{0(0)}} e_{m(q)} \pi^{m(0)} + \frac{e^{0(p)}}{e^{0(0)}} e_k^{(n)} \tilde{\Omega}_{n(pq)}, \tag{40}$$

$$\omega_{m(pq)} = \frac{1}{D-2} \frac{1}{2ee^{0(0)}} \left(e_{m(p)} e_{n(q)} - e_{m(q)} e_{n(p)}\right) \pi^{n(0)} + e_m^{(n)} \tilde{\Omega}_{n(pq)}. \tag{41}$$

This is a linear transformation (in auxiliary fields) from the spatial components of the connections $\omega_{m(\alpha\beta)}$ to the new set of variables $\pi^{m(\rho)}$ and $\tilde{\Omega}_{n(pq)}$.

Note that this field redefinition, (40)–(41), equally well can be performed at the Lagrangian level, in this case "momenta" $\pi^{m(\rho)}$ are just new auxiliary variables that play a role of momenta conjugate to $e_{m(\rho)}$ only after passing to the Hamiltonian formulation. In the Lagrangian formalism, (40)–(41) are a definition of Darboux coordinates; and from now on the auxiliary field $\pi^{m(\rho)}$ will be denoted as $F^{m(\rho)}$.

We can combine (40) and (41) into one "semicovariant" expression

$$\omega_{m(\alpha\beta)} = N_{m(\alpha\beta)0n(\sigma)} F^{n(\sigma)} + e_{p(\alpha)} e_{q(\beta)} e_m^{(n)} \gamma^{p(b)} \gamma^{q(c)} \tilde{\Omega}_{n(bc)} \tag{42}$$

where

$$N_{m(\alpha\beta)0n(\sigma)} = \frac{1}{2ee^{0(0)}} \times \left[\left(\tilde{\delta}_\alpha^q \tilde{\delta}_\beta^0 - \tilde{\delta}_\alpha^0 \tilde{\delta}_\beta^q\right)\left(-I_{m(q)n(p)} \tilde{\delta}_\sigma^p + \frac{D-3}{D-2} e_{m(0)} e_{n(q)} \tilde{\delta}_\sigma^0\right) + \tilde{\delta}_\alpha^p \tilde{\delta}_\beta^q \frac{1}{D-2} \left(e_{m(p)} e_{n(q)} - e_{m(q)} e_{n(p)}\right) \tilde{\delta}_\sigma^0\right]. \tag{43}$$

The advantage of going to Darboux variables

$$L\left(e_{\mu(\alpha)}, \omega_{\mu(\alpha\beta)}\right) \to L\left(e_{\mu(\alpha)}, F^{m(\rho)}, \omega_{0(\alpha\beta)}, \tilde{\Omega}_{n(pq)}\right) \tag{44}$$

is based on the following properties

$$eB^{k(\rho)0(\alpha)m(\beta)}N_{m(\alpha\beta)0n(\sigma)} = \delta_n^k \tilde{\delta}_\sigma^\rho, \qquad B^{k(\rho)0(\alpha)m(\beta)}\omega_{m(\alpha\beta)}\left(\tilde{\Omega}\right) = 0 \tag{45}$$

that for the "kinetic part" of the original Lagrangian (2) gives a simple expression that is quadratic in fields

$$eB^{k(\rho)0(\alpha)m(\beta)}e_{k(\rho),0}\omega_{m(\alpha\beta)} = F^{k(\rho)}e_{k(\rho),0}. \tag{46}$$

The possibility of eliminating the field $\tilde{\Omega}_{c(pq)}$ at the Lagrangian level (Lagrangian reduction) depends on the presence of terms quadratic in this field. The "semicovariant" form (42) makes the calculation quite simple as the only source of a term quadratic in $\tilde{\Omega}_{c(pq)}$ that contributes in (2) is the following (see (21))

$$-eA^{k(\alpha)m(\beta)}\omega_{k(\alpha\gamma)}\omega_m{}^{(\gamma}{}_{\beta)}. \tag{47}$$

Substitution of $\omega_{m(\alpha\beta)}(\tilde{\Omega})$ into (47) and contraction with $A^{k(\alpha)m(\beta)}$ gives

$$L\left(\tilde{\Omega}\tilde{\Omega}\right) = e\tilde{\Omega}_{b(pn)}\tilde{\Omega}^{p(nb)} - e\frac{e^{0(a)}}{e^{0(0)}}\tilde{\Omega}^q{}_{(ap)}\frac{e^{0(b)}}{e^0_{(0)}}\tilde{\Omega}^p{}_{(bq)}. \tag{48}$$

Upon performing a variation with respect to $\tilde{\Omega}$, an equation similar to (R102) follows, which as we demonstrated in [2], can be solved; although the second term of (48) makes the calculations quite long (note that in the Darboux coordinates (42) we have the equation (48) as (R102) immediately, not after long preliminary calculations as in [2]). This suggests an additional change of the Darboux coordinates separately for the part proportional to $\tilde{\Omega}_{b(pn)}$; and the first choice (as we have to keep the number of components the same) is the antisymmetric traceless field $\hat{\Sigma}_m{}^{(pq)}$, but with all indices being external and spatial. Such a field is defined as

$$\hat{\Sigma}_m{}^{(pq)} = e_m^{(n)}\gamma^{p(b)}\gamma^{q(c)}\tilde{\Omega}_{n(bc)}, \qquad \tilde{\Omega}^k{}_{(ab)} = \gamma^{m(k)}e_{p(a)}e_{q(b)}\hat{\Sigma}_m{}^{(pq)}, \tag{49}$$

which is an invertible redefinition of auxiliary fields.

This additional redefinition diagonalizes (48) and that can be checked by substitution of (49) into (48), which leads to only one term that is quadratic in $\hat{\Sigma}_m{}^{(pq)}$

$$L\left(\tilde{\Omega}\tilde{\Omega}\right) \Longrightarrow L\left(\hat{\Sigma}\hat{\Sigma}\right) = eg_{qp}\hat{\Sigma}_m{}^{(kp)}\hat{\Sigma}_k{}^{(mq)} \tag{50}$$

(here g_{qp} is a short-hand notation for $e_{q(\alpha)}e_p^{(\alpha)}$, not an independent field).

This completes the derivation of the Darboux coordinates written down in the preceding section in (14)

$$\omega_{m(\alpha\beta)} = N_{m(\alpha\beta)0n(\sigma)}F^{n(\sigma)} + e_{p(\alpha)}e_{q(\beta)}\hat{\Sigma}_m{}^{(pq)} = \omega_{m(\alpha\beta)}(F) + \omega_{m(\alpha\beta)}\left(\hat{\Sigma}\right). \tag{51}$$

The second property of (45) is unaltered by the change of variables in (49) and

$$B^{k(\rho)0(\alpha)m(\beta)}\omega_{m(\alpha\beta)}\left(\hat{\Sigma}\right) = 0. \tag{52}$$

Using transformation (51) we can now obtain the equivalent Lagrangian in terms of Darboux coordinates, perform a Lagrangian reduction (i.e. eliminate $\hat{\Sigma}$) and find the corresponding Hamiltonian as it was schematically indicated in (13); or equally well, we can start the Hamiltonian formulation using the Lagrangian in Darboux coordinates and perform the Hamiltonian reduction. Of course, using either method, we obtain the same result (4)–(5). But before we write the Lagrangian in Darboux coordinates and the corresponding Hamiltonian we would like to make a few comments.

In their discussion of Darboux coordinates specialized to the $D=4$ case, the authors of [25] emphasize the non-linearity of their transformations. Non-linearity in [25] and in our (14) appears with respect to only the non-second class fields (the tetrads of [25] or N-beins in our case); and conversely, linearity in the second class fields ($\pi^{k(\rho)}$, $\hat{\lambda}_{km}$ in [25] and $F^{n(\sigma)}$, $\hat{\Sigma}_m{}^{(pq)}$ in our case) is the main feature of this transformation, making it invertible, which is a necessary condition to establish equivalence of the original formulation with the formulation in terms of Darboux coordinates.

In constructing of (43) we used the results of the Hamiltonian analysis that preserves the $D=3$ limit. A different, "more covariant", combination can be constructed that also diagonalizes the "kinetic" part of the Lagrangian (i.e. has the same properties as (43)); for example

$$N'_{m(\alpha\beta)0n(\sigma)} = \frac{1}{e}\left[e_{m(\sigma)}A_{n(\alpha)0(\beta)} - \frac{1}{D-2}\left(e_{m(\alpha)}A_{n(\sigma)0(\beta)} - e_{m(\beta)}A_{n(\sigma)0(\alpha)}\right)\right] \tag{53}$$

where

$$A_{n(\alpha)0(\beta)} = e_{n(\alpha)}e_{0(\beta)} - e_{n(\beta)}e_{0(\alpha)}. \tag{54}$$

Without any preliminary Hamiltonian analysis, and working only in a particular dimension (e.g. $D=4$), such a diagonalization of the "kinetic" part looks even preferable as it has a "more covariant" form that simplifies calculations and does not involve a division by $e^{0(0)}$ as in (43). However, it does not have the correct $D=3$ limit, which cannot be seen if one is working only in a particular dimension, e.g. when $D=4$ (see point (C) in the Introduction). Using the Darboux coordinates (51) with $N'_{m(\alpha\beta)0n(\sigma)}$, instead of $N_{m(\alpha\beta)0n(\sigma)}$, leads to problems in the Hamiltonian analysis. The reason for this is that the transformation found using the Hamiltonian analysis, (43), preserves properties of the primary variables, i.e. $\frac{\delta\omega_{m(\alpha\beta)}}{\delta e_{0(\rho)}}=0$, for the Darboux coordinates with $N_{m(\alpha\beta)0n(\sigma)}$, as

$$\frac{\delta}{\delta e_{0(\rho)}}\left(N_{m(\alpha\beta)0n(\sigma)}F^{n(\sigma)} + e_{p(\alpha)}e_{q(\beta)}\hat{\Sigma}_m{}^{(pq)}\right) = 0, \tag{55}$$

in contrast to the "more covariant" combination $N'_{m(\alpha\beta)0n(\sigma)}$ for which

$$\frac{\delta}{\delta e_{0(\rho)}}N'_{m(\alpha\beta)0n(\sigma)} \neq 0. \tag{56}$$

This might lead to change of the algebra of constraints, gauge invariance, etc.[6]

This is an illustration of how the properties of a singular system can be drastically changed even at the Lagrangian level if one assumes that it is always permissible to use some operations (e.g. any invertible transformation) known for non-singular Lagrangians

[6] This example illustrates why problems might arise in the Faddeev-Jackiw method [36]. The symplectic form in [36] was found by diagonalizing the "kinetic" part of the Lagrangian; but, as we show, this is not enough in general to preserve equivalence. And this is the reason for the "non-equivalence" of the Dirac and symplectic methods found for some models.

without careful analysis and without taking into account the specifics of singular systems (see the general discussion in [41]). In constructing the Darboux coordinates we rely on the Hamiltonian analysis and follow Dirac's general statement [4]: "I [Dirac] feel that there will always be something missing from them [non-Hamiltonian methods] which we can only get by working from a Hamiltonian". However, the criteria of equivalence based on the Noether second theorem for singular systems can be formulated at the pure Lagrangian level; the results will be reported elsewhere [42].

4 Lagrangian and Hamiltonian Reductions of the EC Theory in Darboux Coordinates

Substitution of the Darboux coordinates (51) into the original EC Lagrangian L_{EC} (2) is a simple task, as we have only one expression for all spatial connections (51) which are the only fields that are affected by a change of variables. This gives us a different, but equivalent first order formulation of the EC theory $L_{EC(D)}$

$$\begin{aligned}
&L_{EC}\left(e_{\mu(\alpha)},\omega_{\mu(\alpha\beta)}\right) \quad \Longrightarrow \\
&L_{EC(D)}\left(e_{\mu(\alpha)},\omega_{0(\alpha\beta)},F^{k(\rho)},\hat{\Sigma}_m{}^{(pq)}\right) \\
&\quad = e_{k(\rho),0}F^{k(\rho)} + \left(eB^{k(\rho)m(\alpha)0(\beta)}e_{k(\rho),m} - 2eA^{0(\alpha)k(\gamma)}\omega_k{}^{(\beta}{}_{\gamma)}\left(F,\hat{\Sigma}\right)\right)\omega_{0(\alpha\beta)} \\
&\qquad - e_{0(\rho),k}F^{k(\rho)} + eB^{n(\rho)k(\alpha)m(\beta)}e_{n(\rho),k}\omega_{m(\alpha\beta)}\left(F,\hat{\Sigma}\right) \\
&\qquad - eA^{k(\alpha)m(\beta)}\omega_{k(\alpha\gamma)}\left(F,\hat{\Sigma}\right)\omega_m{}^{(\gamma}{}_{\beta)}\left(F,\hat{\Sigma}\right).
\end{aligned} \tag{57}$$

The appearance of two terms that are quadratic in the fields (first terms in the second and third lines of (57)) is the consequence of (46). Further, separating the spatial connections into two parts (see (40), (41)) and performing some contractions we obtain

$$\begin{aligned}
L_{EC(D)} = {} & e_{k(\rho),0}F^{k(\rho)} + \left(\frac{1}{2}F^{k(\alpha)}e_k^{(\beta)} - \frac{1}{2}F^{k(\beta)}e_k^{(\alpha)} + eB^{k(\rho)m(\alpha)0(\beta)}e_{k(\rho),m}\right)\omega_{0(\alpha\beta)} \\
& - e_{0(\rho),k}F^{k(\rho)} + eB^{n(\rho)k(\alpha)m(\beta)}e_{n(\rho),k}\omega_{m(\alpha\beta)}(F) - eA^{k(\alpha)m(\beta)}\omega_{k(\alpha\gamma)}(F)\omega_m{}^{(\gamma}{}_{\beta)}(F) \\
& + eg_{qp}\hat{\Sigma}_m{}^{(kp)}\hat{\Sigma}_k{}^{(mq)} + 2ee^{k(\beta)}\left(e_{m(\beta),q} + e_q^{(\gamma)}\omega_{m(\gamma\beta)}(F)\right)\hat{\Sigma}_k{}^{(mq)},
\end{aligned} \tag{58}$$

where g_{qp} is, as before, a short-hand notation for $e_{q(\rho)}e_p^{(\rho)}$, not an independent field. Note that in the terms proportional to $\omega_{0(\alpha\beta)}$, there are no contributions involving $\hat{\Sigma}_k{}^{(mq)}$ as $A^{0(\alpha)k(\gamma)}\omega_k{}^{(\beta}{}_{\gamma)}(\hat{\Sigma}) = 0$; and contributions with $F^{k(\alpha)}$, instead of direct substitution, can be obtained by contracting $B^{k(\rho)m(\alpha)0(\beta)}$ with $e_k^{(\beta)}$ and performing an antisymmetrization that gives $eA^{0(\alpha)k(\gamma)}\omega_k{}^{(\beta}{}_{\gamma)}(F)$. The last line of (58) is the result of a contraction with the explicit form of $\omega_{k(\alpha\gamma)}(\hat{\Sigma})$ (see (51)).

Equation (58) is the algebraic expression with respect to the field $\hat{\Sigma}_k{}^{(mq)}$ which can be eliminated by using its equation of motion due to the presence of a term in the Lagrangian quadratic in this field (Lagrangian reduction). After elimination of this field we can obtain the Lagrangian (see (11)) with $F^{k(\rho)}$ playing the role of momenta conjugate to $e_{k(\rho)}$ in the Hamiltonian formulation, and without the need to solve the secondary second class constraints. Thus, using the Darboux coordinates we can obtain (4)–(5), which is the same Hamiltonian derived in [2], but without having to do any long calculation. Equivalence of the Lagrangian and Hamiltonian methods allows us to interchange the order of operations, and using (58) (without the Lagrangian reduction) we can immediately write the Hamiltonian and then perform the Hamiltonian reduction. This approach is employed in the present paper, giving us the possibility to compare the results obtained here with the direct calculations of [2].

The Hamiltonian of the first order EC action written in Darboux coordinates can be just read off from the corresponding Lagrangian (58), as it is possible for any first order action. The advantage of Darboux coordinates is that the primary constraints are very simple and the second class variables $F^{k(\alpha)}$ and $\hat{\Sigma}_k{}^{(mq)}$ can be easily separated and eliminated by solving the second class constraints (Hamiltonian reduction).

Note that we can use the Lagrangian (58) and eliminate field $\hat{\Sigma}_k{}^{(mq)}$ without any reference to the Hamiltonian and only after that go to the Hamiltonian, which can be read off from such a reduced Lagrangian. Equally well, we can perform reduction at the Hamiltonian level starting from the Lagrangian (58). We use the second approach to compare, or rather illustrate, simplifications that occur after using the Darboux coordinates with the direct calculations in [2]. In addition, such an approach allows us to discuss some subtle points.

The total Hamiltonian (introducing momenta conjugate to all fields in the Lagrangian (58)) is

$$H_T\left(e_{\mu(\rho)}, \pi^{\mu(\rho)}, \omega_{0(\alpha\beta)}, \Pi^{0(\alpha\beta)}, F^{k(\alpha)}, \Pi_{k(\alpha)}, \hat{\Sigma}_k{}^{(mq)}, \hat{\Pi}^k{}_{(mq)}\right)$$
$$= \pi^{\mu(\rho)}\dot{e}_{\mu(\rho)} + \Pi^{0(\alpha\beta)}\dot{\omega}_{0(\alpha\beta)} + \Pi_{k(\alpha)}\dot{F}^{k(\alpha)} + \hat{\Pi}^k{}_{(mq)}\hat{\Sigma}_{k,0}{}^{(mq)} - L, \tag{59}$$

where $\pi^{\mu(\rho)}$, $\Pi^{0(\alpha\beta)}$, $\Pi_{k(\alpha)}$ and $\hat{\Pi}^k{}_{(mq)}$ are momenta conjugate to $e_{\mu(\rho)}$, $\omega_{0(\alpha\beta)}$, $F^{k(\alpha)}$ and $\hat{\Sigma}_k{}^{(mq)}$, respectively. One can ask what are the relationships between the momenta $\Pi_{k(\alpha)}$, $\hat{\Pi}^k{}_{(mq)}$ introduced here and the momenta $\Pi^{k(\alpha\beta)}$ conjugate to the spatial connection $\omega_{k(\alpha\beta)}$ in the direct approach [2]? Please note that this question can be completely avoided if we perform the Lagrangian reduction. Is a change of variables $\omega_{k(\alpha\beta)}, \Pi^{k(\alpha\beta)} \longrightarrow F^{k(\alpha)}, \Pi_{k(\alpha)}, \hat{\Sigma}_k{}^{(mq)}, \hat{\Pi}^k{}_{(mq)}$ canonical? Is it necessary for this change to be canonical? In the papers that discuss canonical transformations for constraint systems there are statements that variables corresponding to the second class constraints (and so can be eliminated) do not need to satisfy conditions of canonicity and only variables that cannot be eliminated must do (e.g. that correspond to primary constraints). These results probably explain why in some articles on the symplectic approach the variables that are included in the "kinetic" part of the Lagrangian are called "canonical" and those that can be eliminated are called "non-canonical". However, this question seems to us deserves additional investigation because in the Hamiltonian formulation of the first order, affine-metric, gravity [10] we were able to perform such a separation of variables preserving canonicity including variables that correspond to the second class constraints. Is it possible to do the same for the EC action? Can this affect the result? We are planning to address these issues in the near future.

Returning to (58) and separating terms with "velocities" in the Lagrangian, we write

$$-L = -e_{k(\rho),0}F^{k(\rho)} + H_c$$

and then singling out terms proportional to $\omega_{0(\alpha\beta)}$ we obtain

$$H_c = -\omega_{0(\alpha\beta)}\chi^{0(\alpha\beta)} + H_c' \tag{60}$$

where

$$\chi^{0(\alpha\beta)} = \frac{1}{2}F^{k(\alpha)}e_k^{(\beta)} - \frac{1}{2}F^{k(\beta)}e_k^{(\alpha)} + eB^{k(\rho)m(\alpha)0(\beta)}e_{k(\rho),m} \tag{61}$$

and

$$\begin{aligned} H_c' = {} & e_{0(\rho),k}F^{k(\rho)} - eB^{n(\rho)k(\alpha)m(\beta)}e_{n(\rho),k}\omega_{m(\alpha\beta)}(F) + eA^{k(\alpha)m(\beta)}\omega_{k(\alpha\gamma)}(F)\omega_m{}^{(\gamma}{}_{\beta)}(F) \\ & - eg_{qp}\hat{\Sigma}_k{}^{(mq)}\hat{\Sigma}_m{}^{(kp)} - 2ee^{k(\beta)}\left(e_{m(\beta),q} + e_q^{(\gamma)}\omega_{m(\gamma\beta)}(F)\right)\hat{\Sigma}_k{}^{(mq)}. \end{aligned} \tag{62}$$

The non-zero fundamental PBs are

$$\left\{e_{\nu(\sigma)}, \pi^{\mu(\rho)}\right\} = \delta_\nu^\mu\tilde{\delta}_\sigma^\rho, \qquad \left\{\omega_{0(\alpha\beta)}, \Pi^{0(\rho\sigma)}\right\} = \tilde{\Delta}_{(\alpha\beta)}^{(\rho\sigma)}, \qquad \left\{F^{k(\alpha)}, \Pi_{m(\beta)}\right\} = \delta_m^k\tilde{\delta}_\beta^\alpha, \tag{63}$$

$$\left\{\hat{\Sigma}_k{}^{(mq)}, \hat{\Pi}^x{}_{(yz)}\right\} = \hat{I}_{k(yz)}^{x(mq)} = \delta_k^x\hat{\Delta}_{(yz)}^{(mq)} - \frac{1}{D-2}\left(\delta_y^x\hat{\Delta}_{(kz)}^{(mq)} - \delta_z^x\hat{\Delta}_{(ky)}^{(mq)}\right) \tag{64}$$

where

$$\tilde{\Delta}_{(\alpha\beta)}^{(\rho\sigma)} \equiv \frac{1}{2}\left(\tilde{\delta}_\alpha^\rho\tilde{\delta}_\beta^\sigma - \tilde{\delta}_\beta^\rho\tilde{\delta}_\alpha^\sigma\right), \qquad \hat{\Delta}_{(yz)}^{(mq)} \equiv \frac{1}{2}\left(\delta_y^m\delta_z^q - \delta_z^m\delta_y^q\right). \tag{65}$$

As in any first order formulation, the number of primary constraints is equal to the number of independent variables. One pair of primary constraints

$$\phi^{k(\rho)} = \pi^{k(\rho)} - F^{k(\rho)} \approx 0, \qquad \Pi_{m(\gamma)} \approx 0 \tag{66}$$

is second class. These are constraints of a special form [33], and one pair of phase-space variables can be eliminated without affecting the PBs of the remaining variables by substitution of the solution into the total Hamiltonian

$$F^{k(\rho)} = \pi^{k(\rho)}, \qquad \Pi_{m(\gamma)} = 0. \tag{67}$$

This is the first step of Hamiltonian reduction and illustrates the classification of variables (suggested in the previous section) on primary and second class: $F^{k(\rho)}$ is a second class variable. After this reduction the total Hamiltonian is

$$\begin{aligned} H_T = {} & \pi^{0(\rho)}\dot{e}_{0(\rho)} + \Pi^{0(\alpha\beta)}\dot{\omega}_{0(\alpha\beta)} + \hat{\Sigma}_{k,0}{}^{(mq)}\hat{\Pi}^k{}_{(mq)} - \omega_{0(\alpha\beta)}\chi^{0(\alpha\beta)}(F=\pi) \\ & + H_c'(F=\pi). \end{aligned} \tag{68}$$

According to the Dirac procedure, the next step is to consider the time development of the primary constraints ($\Pi^{0(\alpha\beta)}$, $\pi^{0(\rho)}$, $\hat{\Pi}^k{}_{(mq)}$). After the first reduction, all primary constraints obviously have zero PBs among themselves (they are momenta of canonical variables), i.e. there are no second class pairs among the primary constraints, and all of them lead to the corresponding secondary constraints, e.g.

$$\dot{\Pi}^{0(\alpha\beta)} = \left\{\Pi^{0(\alpha\beta)}, H_T\right\} = \left\{\Pi^{0(\alpha\beta)}, H_c\right\} = \left\{\Pi^{0(\alpha\beta)}, -\omega_{0(\alpha\beta)}\chi^{0(\alpha\beta)}\right\} = \chi^{0(\alpha\beta)}. \tag{69}$$

The secondary rotational constraint $\chi^{0(\alpha\beta)}$ has zero PBs with all primary constraints

$$\left\{\chi^{0(\alpha\beta)}, \Pi^{0(\nu\mu)}\right\} = \left\{\chi^{0(\alpha\beta)}, \pi^{0(\sigma)}\right\} = \left\{\chi^{0(\alpha\beta)}, \hat{\Pi}^k{}_{(mq)}\right\} = 0. \tag{70}$$

The first and the last PBs are manifestly zero and the second is zero as a consequence of the properties of $B^{\lambda(\gamma)\mu(\alpha)\nu(\beta)}$ and antisymmetry of $C^{\tau(\sigma)\lambda(\gamma)\mu(\alpha)\nu(\beta)}$ (see (A.3) of Appendix A)

$$\frac{\delta}{\delta e_{0(\sigma)}}\left(eB^{k(\rho)m(\alpha)0(\beta)}e_{k(\rho),m}\right)=eC^{0(\sigma)k(\rho)m(\alpha)0(\beta)}e_{k(\rho),m}=0. \tag{71}$$

The PB among two rotational constraints corresponds to Lorentz algebra (6); this has already been demonstrated in [1, 2] for all dimensions.

Because $\{\pi^{0(\rho)},\chi^{0(\alpha\beta)}\}=0$ and since the PBs among all the primary constraints (69) all vanish, the time development of $\pi^{0(\sigma)}$ leads to the secondary translational constraint

$$\dot{\pi}^{0(\sigma)}=\left\{\pi^{0(\sigma)},H_T\right\}=\left\{\pi^{0(\sigma)},H_c\right\}=\left\{\pi^{0(\sigma)},H'_c\right\}=-\frac{\delta H'_c}{\delta e_{0(\sigma)}}=\chi^{0(\sigma)}, \tag{72}$$

which has the following explicit form

$$\chi^{0(\sigma)}=\pi^{k(\sigma)}_{,k}+eC^{0(\sigma)n(\rho)k(\alpha)m(\beta)}e_{n(\rho),k}\omega_{m(\alpha\beta)}(\pi)-eB^{0(\sigma)k(\alpha)m(\beta)}\omega_{k(\alpha\gamma)}(\pi)\,\omega_m{}^{(\gamma}{}_{\beta)}(\pi)$$
$$+ee^{0(\sigma)}g_{qp}\hat{\Sigma}_k{}^{(mq)}\hat{\Sigma}_m{}^{(kp)}+2eA^{0(\sigma)k(\beta)}\left(e_{m(\beta),q}+e^{(\gamma)}_q\omega_{m(\gamma\beta)}(\pi)\right)\hat{\Sigma}_k{}^{(mq)}. \tag{73}$$

When performing the variation in (72) we used $\frac{\delta\omega_{k(\gamma\beta)}(\pi)}{\delta e_{0(\sigma)}}=0$ (this is easy to show using $\omega_{k(\gamma\beta)}(\pi)$ from (51) and the fact that $\frac{\delta}{\delta e_{0(\sigma)}}(\frac{1}{ee^{0(0)}})=0$ and $\frac{\delta e_{k(\mu)}}{\delta e_{0(\sigma)}}=0$). In all terms in H'_c, only ABC density functions are affected and their variations are simple (see Appendix A).

Contracting (73) with $e_{0(\sigma)}$ and using the ABC properties (expand A and B in σ and contract with $e_{0(\sigma)}$ (see (A.12) of Appendix A)), we can express H'_c as

$$H'_c=-e_{0(\sigma)}\chi^{0(\sigma)}+\left(e_{0(\rho)}\pi^{k(\rho)}\right)_{,k}. \tag{74}$$

Based on the properties of the ABC functions, we immediately obtain the result that the PB of $\chi^{0(\sigma)}$ with the primary translational constraint, $\pi^{0(\mu)}$, is zero (i.e. after second variation with respect to $e_{0(\mu)}$ we will have A, B, C and D with two equal indices (00) which are zero because of antisymmetry of these density functions; see Appendix A):

$$\left\{\chi^{0(\sigma)},\pi^{0(\mu)}\right\}=0, \tag{75}$$

and also

$$\left\{\chi^{0(\sigma)},\Pi^{0(\alpha\beta)}\right\}=0. \tag{76}$$

The PB of $\chi^{0(\sigma)}$ with the primary constraint $\hat{\Pi}^k{}_{(mq)}$ is not zero, but the time development of $\hat{\Pi}^k{}_{(mq)}$ leads to the secondary constraint

$$\hat{\Pi}^k{}_{(mq),0}=\left\{\hat{\Pi}^k{}_{(mq)},H'_c\right\}=-\frac{\delta H'_c}{\delta\hat{\Sigma}_k{}^{(mq)}}=\hat{\chi}^k{}_{(mq)} \tag{77}$$

where

$$\hat{\chi}^k{}_{(mq)}=e\hat{\Sigma}_m{}^{(kp)}g_{pq}-e\hat{\Sigma}_q{}^{(kp)}g_{pm}+e\hat{D}^k{}_{(mq)} \tag{78}$$

with $\hat{D}^k{}_{(mq)}$ being the manifestly antisymmetric and traceless combination (see Appendix B)

$$\hat{D}^k{}_{(mq)}=\hat{D}^k{}_{mq}-\hat{D}^k{}_{qm}-\frac{1}{D-2}\left[\delta^k_m\left(\hat{D}^n{}_{nq}-\hat{D}^n{}_{qn}\right)-\delta^k_q\left(\hat{D}^n{}_{nm}-\hat{D}^n{}_{mn}\right)\right] \tag{79}$$

built from the coefficient which appears in front of terms in (62) that are linear in $\hat{\Sigma}_k{}^{(mq)}$

$$\hat{D}^k{}_{mq} = e^{k(\beta)} e_{m(\beta),q} + e^{k(\beta)} e_q^{(\gamma)} \omega_{m(\gamma\beta)}(\pi). \tag{80}$$

Note that $\hat{D}^k{}_{mq}$ is not antisymmetric or traceless by itself; and (78)–(79) are the result of performing a variation using the fundamental PB of (64).

The pair of constraints $(\hat{\Pi}^k_{(mq)}, \hat{\chi}^x{}_{(yz)})$ is second class because

$$\left\{\hat{\Pi}^k{}_{(mq)}, \hat{\chi}^x{}_{(yz)}\right\} = \hat{N}^{kx}_{(mq)(yz)}, \tag{81}$$

where

$$\hat{N}^{kx}_{(mq)(yz)} = -\frac{e}{2}\Bigg[\delta^k_y\left(g_{zm}\delta^x_q - g_{zq}\delta^x_m\right) - \delta^k_z\left(g_{ym}\delta^x_q - g_{yq}\delta^x_m\right) \\ - \frac{1}{D-2}\left[\delta^x_y\left(g_{zm}\delta^k_q - g_{zq}\delta^k_m\right) - \delta^x_z\left(g_{ym}\delta^k_q - g_{yq}\delta^k_m\right)\right]\Bigg], \tag{82}$$

which is manifestly antisymmetric in (mq) and (yz), traceless in km and kq, in xy and xz, non-zero and not proportional to constraints. (Here g_{zm} again denotes $e_{z(\alpha)}e^{(\alpha)}_m$.) The most important property of $\hat{N}^{kx}_{(mq)(yz)}$ is its invertability and the explicit form of its inverse is given below. The pair of variables $(\hat{\Pi}^k{}_{(mq)}, \hat{\Sigma}_k{}^{(mq)})$ can be eliminated by substitution into the total Hamiltonian

$$\hat{\Pi}^k{}_{(mq)} = 0, \qquad \hat{\Sigma}_z{}^{(mk)} = \frac{1}{2}\left(\gamma^{kn}\hat{D}^m{}_{(nz)} - \gamma^{mn}\hat{D}^k{}_{(nz)} - \gamma^{ky}\gamma^{mw}g_{zx}\hat{D}^x{}_{(wy)}\right), \tag{83}$$

where $\hat{\Sigma}_z{}^{(mk)}$ is the solution of the constraint (78) $\hat{\chi}^k{}_{(mq)} = 0$ (see Appendix B). This again illustrates our classification: $\hat{\Sigma}_k{}^{(mq)}$ is a second class variable as is $F^{k(\rho)}$. In (83) we use a short-hand notation which was originally introduced by Dirac [43] for the Hamiltonian formulation of the Einstein-Hilbert action: $\gamma^{kn} \equiv g^{kn} - \frac{g^{0k}g^{0n}}{g^{00}}$ where $g^{\mu\nu} = e^{\mu(\alpha)}e^{\nu}_{(\alpha)}$.

The elimination of the phase-space pair $(F^{k(\rho)}, \Pi_{k(\rho)})$ by solving the corresponding second class constraint was simple as they are of a special form and it is known that in such a case the Dirac brackets (DBs) of the remaining fields coincide with their original PBs [33]. The pair of second class constraints $(\hat{\Pi}^k{}_{(mq)}, \hat{\chi}^x{}_{(yz)})$ is more complicated and the effect of their elimination on the PBs among the remaining canonical variables has to be checked. In [2] the elimination of $\omega_{k(\alpha\beta)}$ was performed using a different and complicated approach because of the mixture of different components ($\omega_{k(pq)}$ and $\omega_{k(p0)}$) in the equations; and it was even not clear how DBs can be calculated. After introducing Darboux coordinates and decoupling the two fields, $F^{k(\rho)}$ and $\hat{\Sigma}_k{}^{(mq)}$, this becomes possible. Let us investigate the effect of their elimination on the DBs of the remaining fields.

The Dirac bracket is defined for any pair of functions of canonical variables as [4]

$$\{\Phi,\Psi\}_{DB} = \{\Phi,\Psi\}_{PB} - \left(\left\{\Phi,\hat{\Pi}^k{}_{(mq)}\right\}_{PB} \quad \left\{\Phi,\hat{\chi}^k{}_{(mq)}\right\}_{PB}\right) M^{-1}\begin{pmatrix}\{\hat{\Pi}^a{}_{(bc)},\Psi\}_{PB}\\ \{\hat{\chi}^a{}_{(bc)},\Psi\}_{PB}\end{pmatrix}. \tag{84}$$

(Here we used our set of second class constraints.) M^{-1} is the inverse of the matrix M built from the PBs of the second class constraints

$$M = \begin{bmatrix}\{\hat{\Pi}^k{}_{(mq)},\hat{\Pi}^a{}_{(bc)}\} & \{\hat{\Pi}^k{}_{(mq)},\hat{\chi}^a{}_{(bc)}\}\\ \{\hat{\chi}^k{}_{(mq)},\hat{\Pi}^a{}_{(bc)}\} & \{\hat{\chi}^k{}_{(mq)},\hat{\chi}^a{}_{(bc)}\}\end{bmatrix} = \begin{bmatrix}0 & \hat{N}^{ka}_{(mq)(bc)}\\ -\hat{N}^{ak}_{(bc)(mq)} & \hat{X}^{ka}_{(mq)(bc)}\end{bmatrix}. \tag{85}$$

Note that $\hat{N}^{ka}_{(mq)(bc)} = \hat{N}^{ak}_{(bc)(mq)}$ (see (82)).

We want to investigate the effect of eliminating the second class constraints on the properties of the remaining canonical variables and on the PBs among the functions constructed from them. $\hat{N}^{ka}_{(mq)(bc)}$ is given by (82), and the explicit form of $\hat{X}^{ka}_{(mq)(bc)}$ can be calculated, but the inverse of M can be defined for any $\hat{X}$. The explicit form of $\hat{X}$ is needed only for calculation of DB among two translational constraints; but we do not discuss this in this article. The inverse of such a matrix can be immediately found if the inverse of the off-diagonal blocks is known

$$\hat{N}^{ka}_{(mq)(bc)} \left(\hat{N}^{-1}\right)^{(bc)(yz)}_{ax} = \hat{I}^{k(yz)}_{x(mq)} \tag{86}$$

where $\hat{I}^{k(yz)}_{x(mq)}$ is the fundamental PB for antisymmetric and traceless canonical pair defined in (64). The inverse $(\hat{N}^{-1})^{(bc)(yz)}_{ax}$ is

$$\begin{aligned}\left(\hat{N}^{-1}\right)^{(bc)(yz)}_{ax} = \frac{1}{4e}\bigg[& g_{ax}\left(\gamma^{by}\gamma^{cz} - \gamma^{bz}\gamma^{cy}\right) + \delta^b_x\left(\gamma^{cz}\delta^y_a - \gamma^{cy}\delta^z_a\right) - \delta^c_x\left(\gamma^{bz}\delta^y_a - \gamma^{by}\delta^z_a\right)\\ &- \frac{2}{D-2}\left[\delta^b_a\left(\gamma^{cz}\delta^y_x - \gamma^{cy}\delta^z_x\right) - \delta^c_a\left(\gamma^{bz}\delta^y_x - \gamma^{by}\delta^z_x\right)\right]\bigg].\end{aligned} \tag{87}$$

$(\hat{N}^{-1})^{(bc)(yz)}_{ax}$ is also manifestly antisymmetric in (bc) and (yz), as well it is traceless in ba and ca, in yx and xz. The following properties are useful

$$\hat{I} \cdot \hat{I} = \hat{I}, \qquad \hat{I} \cdot \hat{N} = \hat{N}, \qquad \hat{I} \cdot \hat{N}^{-1} = \hat{N}^{-1}. \tag{88}$$

Equation (86) and properties (88) allow us to find M^{-1}

$$M^{-1} = \begin{bmatrix} \hat{N}^{-1}\hat{X}\hat{N}^{-1} & -\hat{N}^{-1} \\ \hat{N}^{-1} & 0 \end{bmatrix}, \qquad M \cdot M^{-1} = \begin{bmatrix} \hat{I} & 0 \\ 0 & \hat{I} \end{bmatrix}. \tag{89}$$

The canonical variables which remain after elimination of the pair $(\hat{\Pi}^z_{(mk)}, \hat{\Sigma}_z{}^{(mk)})$ have fundamental PBs that are not affected, as can be easily checked. Substitution of the solution of the secondary constraints (83) into (68) (Hamiltonian reduction) leads to

$$\begin{aligned}H_T = {}& \pi^{0(\rho)}\dot{e}_{0(\rho)} + \Pi^{0(\alpha\beta)}\dot{\omega}_{0(\alpha\beta)} - \omega_{0(\alpha\beta)}\chi^{0(\alpha\beta)}\\ &- e_{0(\sigma)}\chi^{0(\sigma)} \quad \left(\hat{\Sigma}_z{}^{(mk)} \text{ from } (83)\right).\end{aligned} \tag{90}$$

After substitution of $\hat{\Sigma}_z{}^{(mk)}$, this leads to the same result as obtained in [2]. One can see that the Darboux coordinates significantly simplify the calculations.

The main goal of this paper is the construction of Darboux coordinates for the EC action in a form independent of a particular dimension. The direct approach used in [2] made further calculations almost unmanageable; and the simplification due to Darboux coordinates that shortens the calculations of (90), gives us a hope of completing the Dirac procedure. We wish to demonstrate its closure, the absence of tertiary constraints, and restore gauge invariance. This result will be reported elsewhere. Here we just demonstrate that with Darboux coordinates these calculations are drastically simplified; and as an example, we consider the PB between rotational and translational constraints. We argued in [2] that the known invariance of the EC action under Lorentz rotation, Dirac's conjecture [4] and the Castellani algorithm [14] lead to the necessity of having the PB among rotational and translational constraints be exactly the same in all dimensions given by the corresponding part of the Poincaré algebra (9).

Let us demonstrate that indeed in all dimensions ($D > 2$) the PB among translational and rotational constraints is the same and corresponds to the Poincaré algebra (9). This part of algebra among the secondary constraints $\{\chi^{0(\sigma)}, \chi^{0(\mu\nu)}\}$, as well as $\{\chi^{0(\sigma)}, \chi^{0(\rho)}\}$, can be calculated using DBs, i.e. avoiding substitution of $\hat{\Sigma}$ into the translational constraint before calculating the PB (which is the longest part of such calculations) and performing this substitution only after

$$\left\{\chi^{0(\sigma)}, \chi^{0(\mu\nu)}\right\}_{DB} = \left\{\chi^{0(\sigma)}, \chi^{0(\mu\nu)}\right\} - \left(\left\{\chi^{0(\sigma)}, \hat{\Pi}^k{}_{(mq)}\right\} \left\{\chi^{0(\sigma)}, \hat{\chi}^k{}_{(mq)}\right\}\right) M^{-1} \begin{pmatrix} \{\hat{\Pi}^a{}_{(bc)}, \chi^{0(\mu\nu)}\} \\ \{\hat{\chi}^a{}_{(bc)}, \chi^{0(\mu\nu)}\} \end{pmatrix}, \tag{91}$$

where $\hat{\Sigma}$ and $\hat{\Pi}$ are the fundamental variables and the only non-zero PBs are given in (63)–(64). (Note that only the first step of reduction is performed in (67).) After calculating (91) the solution of $\hat{\Sigma}$ is substituted that gives us the final answer for $\{\chi^{0(\sigma)}, \chi^{0(\mu\nu)}\}$. The advantage of this calculation lies in the possibility to demonstrate (and also single out) contributions in the first term of (91) that give the corresponding part of the Poincaré algebra almost manifestly and in compact form. First of all, we outline the idea of such calculations. It is not difficult to demonstrate that the calculation of the first PB in (91) gives

$$\left\{\chi^{0(\sigma)}, \chi^{0(\mu\nu)}\right\} = \frac{1}{2}\tilde{\eta}^{\sigma\mu}\chi^{0(\nu)}\left(\pi, e, \hat{\Sigma}\right) - \frac{1}{2}\tilde{\eta}^{\sigma\nu}\chi^{0(\mu)}\left(\pi, e, \hat{\Sigma}\right) + R^{\sigma(\mu\nu)}\left(\hat{\Sigma}, \pi, e\right). \tag{92}$$

So, the substitution of the solution of $\hat{\Sigma}$ will not affect the first two terms in (92). What is left is to demonstrate that the remainder $R^{\sigma(\mu\nu)}$, along with the second contribution in (91), after substitution of $\hat{\Sigma}$ gives zero which is a long but straightforward calculation. It is easier to prove that $R^{\sigma(\mu\nu)}$ is zero if we consider separately the terms of different "nature"; for example, all terms which are linear in momenta should cancel independently of the rest of contributions. This allows us to break these long and cumbersome calculations into smaller and independent pieces.

Let us outline the proof of (92). We start the calculations by separating the contributions of different order in $\hat{\Sigma}$ in the translational constraint (73)

$$\chi^{0(\sigma)} = \chi^{0(\sigma)}\left(\hat{\Sigma}^2\right) + \chi^{0(\sigma)}\left(\hat{\Sigma}^1\right) + \chi^{0(\sigma)}\left(\hat{\Sigma}^0\right). \tag{93}$$

For the contribution quadratic in $\hat{\Sigma}$, we almost immediately obtain the exact expression (there are no contributions into the remainder in this order, $R^{\sigma(\mu\nu)}(\hat{\Sigma}^2) = 0$)

$$\left\{\chi^{0(\sigma)}\left(\hat{\Sigma}^2\right), \chi^{0(\mu\nu)}\right\} = \frac{1}{2}\tilde{\eta}^{\sigma\mu}\chi^{0(\nu)}\left(\hat{\Sigma}^2\right) - \frac{1}{2}\tilde{\eta}^{\sigma\nu}\chi^{0(\mu)}\left(\hat{\Sigma}^2\right). \tag{94}$$

The next contribution, linear in $\hat{\Sigma}$, is

$$\left\{\chi^{0(\sigma)}\left(\hat{\Sigma}\right), \chi^{0(\mu\nu)}\right\} = \left(e_{m(\rho),q} + e_q^{(\gamma)}\omega_{m(\gamma\rho)}(\pi)\right)\hat{\Sigma}_k{}^{(mq)}\left\{2eA^{0(\sigma)k(\rho)}, \frac{1}{2}\pi^{n(\mu)}e_n^{(\nu)} - \frac{1}{2}\pi^{n(\nu)}e_n^{(\mu)}\right\}$$

$$+ 2eA^{0(\sigma)k(\rho)}\hat{\Sigma}_k{}^{(mq)}\left\{\left(e_{m(\rho),q} + e_q^{(\gamma)}\omega_{m(\gamma\rho)}(\pi)\right),\right.$$
$$\left.\frac{1}{2}\pi^{n(\mu)}e_n^{(\nu)} - \frac{1}{2}\pi^{n(\nu)}e_n^{(\mu)} + eB^{n(\rho)m(\mu)0(\nu)}e_{n(\rho),m}\right\}. \tag{95}$$

Considering the PB in the first term of (95) and using (A.2) (see Appendix A), we obtain

$$\left\{2eA^{0(\sigma)k(\rho)}, \frac{1}{2}\pi^{n(\mu)}e_n^{(\nu)} - \frac{1}{2}\pi^{n(\nu)}e_n^{(\mu)}\right\} = eB^{n(\mu)0(\sigma)k(\rho)}e_n^{(\nu)} - (\mu \leftrightarrow \nu). \tag{96}$$

Expanding B in n (see (A.6) of Appendix A) and contracting it with $e_n^{(\nu)}$ we have

$$eB^{n(\mu)0(\sigma)k(\rho)}e_n^{(\nu)} = e\left(\tilde{\eta}^{\mu\nu} - e^{0(\mu)}e_0^{(\nu)}\right)A^{0(\sigma)k(\rho)} + e\left(\tilde{\eta}^{\sigma\nu} - e^{0(\sigma)}e_0^{(\nu)}\right)A^{0(\rho)k(\mu)}$$
$$+ e\left(\tilde{\eta}^{\rho\nu} - e^{0(\rho)}e_0^{(\nu)}\right)A^{0(\mu)k(\sigma)}. \tag{97}$$

Three terms in (97) proportional to $e_0^{(\nu)}$ give us

$$-ee_0^{(\nu)}\left(e^{0(\mu)}A^{0(\sigma)k(\rho)} + e^{0(\sigma)}A^{0(\rho)k(\mu)} + e^{0(\rho)}A^{0(\mu)k(\sigma)}\right).$$

The expression in the brackets exactly coincides with expansion of $B^{0(\mu)0(\sigma)k(\rho)}$ (see (A.6) of Appendix A), which automatically equals zero because of antisymmetry of B in external indices (for details see Appendix A). After antisymmetrization of the remaining terms of (96) we have

$$\left\{2eA^{0(\sigma)k(\rho)}, \frac{1}{2}\pi^{n(\mu)}e_n^{(\nu)} - \frac{1}{2}\pi^{n(\nu)}e_n^{(\mu)}\right\}$$
$$= -e\tilde{\eta}^{\nu\sigma}A^{0(\rho)k(\mu)} - e\tilde{\eta}^{\nu\rho}A^{0(\mu)k(\sigma)} + e\tilde{\eta}^{\mu\sigma}A^{0(\rho)k(\nu)} + e\tilde{\eta}^{\mu\rho}A^{0(\nu)k(\sigma)}. \tag{98}$$

The first and third terms of the second line of (98), contracted with the expression in front of the PB in (95), gives exactly two rotational constraints; and the rest of terms, along with the second term in (95), contribute to the remainder. Finally,

$$\left\{\chi^{0(\sigma)}\left(\hat{\Sigma}^1\right), \chi^{0(\mu\nu)}\right\} = \frac{1}{2}\tilde{\eta}^{\sigma\mu}\chi^{0(\nu)}\left(\hat{\Sigma}^1\right) - \frac{1}{2}\tilde{\eta}^{\sigma\nu}\chi^{0(\mu)}\left(\hat{\Sigma}^1\right) + R^{\sigma(\mu\nu)}\left(\hat{\Sigma}^1\right). \tag{99}$$

Similarly one can demonstrate (using properties of A, B, C functions) that in the last order (zero order in $\hat{\Sigma}$) the same result as (99) follows (with the additional contributions into the remainder $R^{\sigma(\mu\nu)}(\hat{\Sigma}^0)$) leading to (92). This part of the calculation is simple and it is needed to complete the proof of (92). For the rest of calculations, the remainder has to be considered together with the second term of (91), and the solution of $\hat{\Sigma}$ has to be substituted at the end of the calculations.

From the field content of $\chi^{0(\mu\nu)}$ (independence from $\hat{\Pi}^a{}_{(bc)}$) it follows that $\{\hat{\Pi}^a{}_{(bc)}, \chi^{0(\mu\nu)}\} = 0$. Using this PB and the explicit form of M^{-1} we obtain

$$\left(\left\{\chi^{0(\sigma)}, \hat{\Pi}^k{}_{(mq)}\right\}\left\{\chi^{0(\sigma)}, \hat{\chi}^k{}_{(mq)}\right\}\right)M^{-1}\begin{pmatrix}\{\hat{\Pi}^a{}_{(bc)}, \chi^{0(\mu\nu)}\}\\ \{\hat{\chi}^a_{(bc)}, \chi^{0(\mu\nu)}\}\end{pmatrix}$$
$$= \left\{\chi^{0(\sigma)}, \hat{\Pi}^k{}_{(mq)}\right\}\left(-\hat{N}^{-1}\right)_{ka}^{(mq)(bc)}\left\{\hat{\chi}^a{}_{(bc)}, \chi^{0(\mu\nu)}\right\}. \tag{100}$$

When calculating (100), it is better to extract terms proportional to $\hat{\chi}^a{}_{(bc)}$, which after substitution of $\hat{\Sigma}$ vanish and we are left with the simple expressions

$$\left\{\chi^{0(\sigma)}, \hat{\Pi}^k{}_{(mq)}\right\} = -2ee^{n(\sigma)}e^{0(\beta)}\left(e_{p(\beta),d} + e_d^{(\gamma)}\omega_{p(\gamma\beta)}(\pi)\right)\left\{\hat{\Sigma}_n{}^{(pd)}, \hat{\Pi}^k{}_{(mq)}\right\} \tag{101}$$

and

$$\left\{\hat{\chi}^a{}_{(bc)}, \chi^{0(\mu\nu)}\right\} = -e\left\{\hat{D}^a{}_{(bc)}, \chi^{0(\mu\nu)}\right\}. \tag{102}$$

Using (101), (102) together with (91), (88), and the antisymmetry and tracelessness of (87) and (79), we obtain

$$\begin{aligned}\left\{\chi^{0(\sigma)}, \chi^{0(\mu\nu)}\right\} &= \frac{1}{2}\tilde{\eta}^{\sigma\mu}\chi^{0(\nu)} - \frac{1}{2}\tilde{\eta}^{\sigma\nu}\chi^{0(\mu)} + R^{\sigma(\mu\nu)} \\ &\quad + 4ee^{n(\sigma)}e^{0(\beta)}\left(e_{p(\beta),b} + e_b^{(\gamma)}\omega_{p(\gamma\beta)}(\pi)\right)\left(\hat{N}^{-1}\right)_{na}^{(pd)(bc)} \\ &\quad \times e\left\{e^{a(\alpha)}e_{b(\alpha),c} + e^{a(\alpha)}e_c^{(\rho)}\omega_{b(\rho\alpha)}(\pi), \chi^{0(\mu\nu)}\right\}.\end{aligned} \tag{103}$$

The most laborious part of the calculation is a demonstration that the remainder, together with the last line of (103), equals zero. We perform these calculations by separating terms of different order in $\pi^{n(\mu)}$, which makes the analysis more manageable.

Using Darboux coordinates allows us to prove that the PB among rotational and translational constraints also supports the Poincaré algebra in all dimensions, $D > 2$. Knowledge of this PB, along with (6), is sufficient to restore rotational invariance in the Hamiltonian formulation of the Einstein-Cartan action by using the Castellani procedure. This result, as well as calculation of the PB between two translational constraints and the restoration of translational invariance, will be reported elsewhere.

5 Discussion

Based on the results of direct application of the Dirac procedure to the first order Einstein-Cartan action [2], we have constructed the uniform Darboux coordinates valid in all dimensions for which the first order formulation exists; i.e. when it is equivalent to the second order EC action ($D > 2$). In particular, these uniform Darboux coordinates guarantee equivalence and allow one to check the $D = 3$ limit [1] at all stages of calculation in dimensions $D > 3$. Considerable simplification occurs when we use Darboux coordinates and it is explicitly demonstrated by obtaining the Hamiltonian formulation in a few lines compared with the direct and cumbersome calculations [2] considered previously. However, we have to emphasize that the preliminary Hamiltonian analysis is indispensable for the construction of Darboux coordinates, which preserve equivalence with the original action. An arbitrary change of variables at the Lagrangian level for singular Lagrangians is an ambiguous operation because it might correspond to a non-canonical change of variables at the Hamiltonian level. For singular Lagrangians the invertability of transformations (redefinition of fields) from one set of variables to another is not a sufficient condition to preserve equivalence [41]; and one particular example is considered in the end of Sect. 3 (see (53)). These "Darboux coordinates", (53), despite separating variables in the same way, do not preserve the $D = 3$ limit, lead to results which are different from those found by the direct analysis and destroy equivalence. Our Darboux transformations, (43), do not suffer such an ambiguity because they are based on the preliminary Hamiltonian analysis. In other words, our transformation separates variables in the same way that the Hamiltonian reduction does. This makes this transformation unique and preserves equivalence with the original action, as well as the equivalence of results for the Lagrangian and Hamiltonian formulations.

To answer the question about possible modifications of the Poincaré algebra of PBs among the secondary constraints of the EC Hamiltonian in dimensions $D > 3$, we need to complete calculations of PBs. In particular, if $\{\chi^{0(\sigma)}, \chi^{0(\mu\nu)}\} = \frac{1}{2}\tilde{\eta}^{\sigma\mu}\chi^{0(\nu)} - \frac{1}{2}\tilde{\eta}^{\sigma\nu}\chi^{0(\mu)}$ (and there is a strong indication that this is the case) and $\{\chi^{0(\alpha)}, \chi^{0(\beta)}\} = 0$, then the N-bein gravity is the Poincaré gauge theory in all dimensions and the $D = 3$ case is not special at all. Note (we discussed this in [2]) that in higher dimensions, the constraints are much more complicated and having the same algebra does not mean that the gauge transformations must be exactly the same as for $D = 3$. If $\{\chi^{0(\alpha)}, \chi^{0(\beta)}\} \neq 0$, but proportional to secondary first class constraints, we still have closure of the Dirac procedure, all constraints are first class, and the gauge generators can be found and the gauge transformations can be restored. In this case, N-bein gravity for $D > 3$ is the gauge theory, but with the modified Poincaré algebra. For example, if (the most general case)

$$\left\{\chi^{0(\alpha)}, \chi^{0(\beta)}\right\} = \tilde{F}^{(\alpha\beta)}_{(\mu\nu)}\chi^{0(\mu\nu)} + \tilde{M}^{(\alpha\beta)}_{\nu}\chi^{0(\nu)} \tag{104}$$

with structure functions $\tilde{F}^{(\alpha\beta)}_{(\mu\nu)}$ and $\tilde{M}^{(\alpha\beta)}_{\nu}$, which are $\tilde{F}^{(\alpha\beta)}_{(\mu\nu)} = 0$ and $\tilde{M}^{(\alpha\beta)}_{\nu} = 0$ when $D = 3$, then one can say that in all dimensions the EC theory is a gauge theory with a generalized Poincaré algebra among secondary first class constraints that degenerates into the true Poincaré algebra when $D = 3$. A similar result as (104) has been known for a long time and was presented in [11, 12, 18]; but it was not obtained using the Hamiltonian procedure and it was written for generators, not for the PBs of constraints. The complete Hamiltonian analysis will show whether the algebra among constraints is Poincaré or modified Poincaré. One can expect some similarities of (104) with the results of the Lagrangian approach. The commutator of two translations is proportional to rotation and translation [3]

$$(\delta_{t''}\delta_{t'} - \delta_{t'}\delta_{t''})\, e_{\nu(\beta)} = \delta_{\tilde{r}} e_{\nu(\beta)} + \delta_{\tilde{t}} e_{\nu(\beta)} \tag{105}$$

with the parameters

$$\tilde{r}_{(\alpha\beta)} \equiv e^{\lambda(\rho)} e^{\tau(\gamma)} R_{\lambda\tau(\alpha\beta)} t'_{(\rho)} t''_{(\gamma)} \tag{106}$$

and

$$\tilde{t}_{(\sigma)} \equiv e^{\lambda(\rho)} e^{\tau(\gamma)} T_{\lambda\tau(\sigma)} t'_{(\rho)} t''_{(\gamma)}. \tag{107}$$

(The same relations (105)–(107) hold for $(\delta_{t''}\delta_{t'} - \delta_{t'}\delta_{t''})\omega_{\nu(\alpha\beta)}$.) Some relationship among the structure functions in (104) and (105)–(107) should be expected.

Of course, the structure functions $\tilde{F}^{(\alpha\beta)}_{(\mu\nu)}$ and $\tilde{M}^{(\alpha\beta)}_{\nu}$ in the Hamiltonian approach will be complicated as part of the variables were eliminated in the course of reduction (solving second class constraints) and the rest of variables and momenta are not present in a manifestly covariant form.

The exact form of the algebra of secondary constraints is important; but we already have enough evidence to make a conclusion about the gauge invariance of the EC action. It is certain that *gauge* invariance is the translation and Lorentz rotation in the internal space and that diffeomorphism (either spatial or full) is not a *gauge invariance* of N-bein gravity generated by its first class constraints. This conclusion is based on the following arguments.

The parameters characterizing the gauge transformations are defined by the tensorial character of primary first class constraints. Both of them, $\Pi^{0(\alpha\beta)}$ and $\pi^{0(\sigma)}$, have internal indices, so do the corresponding gauge parameters $r_{(\alpha\beta)}$ and $t_{(\sigma)}$ (see Sect. V of [10] for more details). This gauge symmetry corresponds to rotation and translation in the tangent space. The gauge parameter of diffeomorphism, ξ^{μ}, has an external index, which can be accommodated only if the corresponding primary first class constraint also has an external free index. This does not happen in the case of N-bein gravity if the Dirac procedure is performed correctly and a non-canonical change of variables is avoided (see [10]). Formulations that claim to have the "spatial diffeomorphism constraint" or any further consideration based on such a constraint [8, 9] for tetrad gravity is the product of non-canonical change of variables, which has the same origin as in metric gravity.[7] Gauge invariance is a unique characteristic of a singular system and follows from its unique constraint structure, i.e. it is derivable from the first class constraints in accordance with Dirac's conjecture [4] and using the Castellani algorithm [14] (the only algorithm to restore gauge invariance which is not sensitive to a choice of combinations of non-primary constraints [10]). Gauge invariance is unique, but it does not presume the absence of additional symmetries in the action. In [13] such symmetries, which are not derivable from constraints, are called "trivial"[8] but this name seems to us to be a little bit confusing as e.g. the non-gauge symmetry of the EC action, diffeomorphism, can hardly be called "trivial". It is more preferable, without introducing any new terminology, to just have "symmetries of the action" (there could be many) and "a gauge symmetry" (a unique one) that follows from the Hamiltonian analysis or from basic differential identities at the Lagrangian level [3].

The Hamiltonian analysis allows us to single out "what is a gauge symmetry and what is not" [15]. In [3] we showed, using differential identities, that the translation in the internal space is an invariance of the EC action. This fact has long been known; such transformations were written in [12, 18], and are exactly the same as we obtained in [3]. This makes the common statement that "translation is not invariance" absolutely groundless and somewhat mysterious. In [3] we also argue that two invariances, translation in the internal space and diffeomorphism, cannot be simultaneously present as *gauge* invariances in Hamiltonian formulations as the number of first class constraints needed to generate both of them would lead to a negative number of degrees of freedom (for relation between the number of constraints and degrees of freedom see [21]). The only possible way to reconcile these two symmetries, as we stated in [3], is that there exists a canonical transformation that converts our constraints (90) into a different set of constraints which support diffeomorphism. In this article we argue that this is impossible and such a canonical transformation does not exist. Let us look at this from the Hamiltonian and Lagrangian points of view.

From the Hamiltonian point of view, the known canonical transformations for the first and second order Einstein-Hilbert actions [10, 44, 45] always preserve the form-invariance of the constraint algebra that would be destroyed by any transformation that changes the tensorial character of the primary constraints needed to have diffeomorphism as a *gauge* invariance of the EC action (as the gauge parameter of diffeomorphism ξ^{μ} is a true vector; for more details see [10]). In addition, in formulations which are related by a canonical

[7]The loss of full diffeomorphism invariance due to a non-canonical change of variables in metric gravity was discussed in [16, 17, 44]. A lapse with canonicity leads to big (or rather a devastating) shift from covariant General Relativity (Einstein-Hilbert and Einstein-Cartan actions) to some non-covariant models like "geometrodynamics" for metric gravity and inspired by it non-covariant models for tetrads (see Sect. V of [10] for discussion on this topic). This, of course, propagates into further analysis (i.e. quantization) of these models.

[8]In Sect. 3.1.5. of [13] "trivial gauge transformations" are defined and all transformations are classified by using the Hamiltonian method; in addition, it is stated that these "transformations are of no physical significance because in the Hamiltonian formalism they are not generated by a constraint".

transformation, constraints are different, but the gauge transformations are the same and for new and old fields and they can be obtained one from another without any need for a field dependent redefinition of gauge parameters (such a field dependent redefinition is an indication of having a non-canonical change of variables). The gauge parameters which are responsible for diffeomorphism, ξ^{μ}, and translation, $t_{(\sigma)}$, cannot be related without involving fields as they have a different tensorial "nature". Therefore, the field dependent redefinition of parameters is needed. Consequently, there is no canonical transformation between such formulations which give diffeomorphism and translation as gauge symmetries. And so, the Hamiltonian formulation with the constraints that would produce diffeomorphism invariance is not equivalent to the Hamiltonian formulation with translation in the tangent space as a gauge symmetry.

From the Lagrangian point of view, using the 16 components of the tetrads (in the $D = 4$ case) we can restore the 10 components of the metric tensor but "not vice versa" [46]. Tetrads are "world" vectors and are invariant under diffeomorphism, as any vector or tensor would be in a generally covariant theory. From the diffeomorphism invariance of tetrads we can derive invariance under diffeomorphism for any combination of tetrads, in particular, for $e_{\mu}^{(\alpha)} e_{\nu(\alpha)} = g_{\mu\nu}$ we obtain

$$\begin{aligned}\delta_{diff}\left(e_{\mu}^{(\alpha)} e_{\nu(\alpha)}\right) &= e_{\mu}^{(\alpha)}\left(-e_{\rho(\alpha)}\xi^{\rho}_{,\nu} - e_{\nu(\alpha),\rho}\xi^{\rho}\right) + \left(-e_{\rho}^{(\alpha)}\xi^{\rho}_{,\mu} - e_{\mu,\rho}^{(\alpha)}\xi^{\rho}\right) e_{\nu(\alpha)} \\ &= -\xi_{\mu,\nu} + g_{\rho\mu,\nu}\xi^{\rho} - \xi_{\nu,\mu} + g_{\rho\nu,\mu}\xi^{\rho} - g_{\mu\nu,\rho}\xi^{\rho} = \delta_{diff} g_{\mu\nu}. \qquad (108)\end{aligned}$$

Note that in this case there is no need for a field dependent redefinition of gauge parameters.

We can also perform the inverse operation: from the diffeomorphism of the metric tensor (which is the gauge invariance of Einstein-Hilbert (EH) action [10, 16, 17, 44]) we can derive the diffeomorphism of tetrads (more details of this derivation and discussion about gauge symmetries of the metric tensor and tetrads are given in [1]). But we cannot obtain Lorentz or translational invariances in the tangent space of the EC action from the diffeomorphism of the metric tensor. The reason for this is simply that the EC and EH actions are not equivalent and neither are the corresponding Hamiltonians: they have a different number of phase-space variables, different constraints, PB algebras, tensorial dimension of primary constraints and different gauge invariances.

In the Lagrangian formalism, if we perform a change of variables that keep the equivalence of two formulations, build differential identities and restore the invariance of each formulation, then the invariance of one formulation must be derivable from the invariance of another, using the same original redefinition of fields and without a redefinition of parameters. If a field dependent redefinition of parameters is needed, then the change of variables that was performed is not canonical at the Hamiltonian level. Hence, such a change of variables should also be checked at the Lagrangian level: to test whether the formulation in the new variables is equivalent to the original one. As an illustration of this, we again compare metric General Relativity (GR) and ADM gravity. For metric GR the Hamiltonian [16, 17] and Lagrangian [47] methods give the same gauge transformation, diffeomorphism. For the ADM gravity the Hamiltonian and Lagrangian methods also produce the same invariance, but one which is different from diffeomorphism (compare [48] and [49]). This is consistent with the fact that the Hamiltonian and Lagrangian approaches give equivalent descriptions of the same system. However, it is clear from [48] and [49], that the gauge transformations of ADM gravity are not diffeomorphism. It is not a surprise as ADM gravity is not equivalent to GR (see [10, 17]).[9] Only after a field dependent redefinition of parameters is performed, is it possible to find an "equivalence between diffeomorphism and gauge transformations" of ADM gravity [48]. The same redefinition is also needed at the Lagrangian level [49], which according to the authors, demonstrates "the equivalence between the gauge and diff parameters by devising of the one to one mapping". The same holds for the EC gravity: the field dependent redefinition of parameters is needed to relate translation and diffeomorphism, so only non-canonical transformations can relate two such Hamiltonian formulations.

An additional argument to support our point of view is related to differential identities from which the invariances of a Lagrangian can be found. As we showed in [3], considering the EC Lagrangian as an example, all differential identities can be constructed from a few basic differential identities. One out of the many identities leading to invariances of the EC action, a *gauge* identity can be singled out using the following arguments. Differential identities leading to *gauge* invariances for known theories are always the simplest: they are built *starting* by contracting derivatives (∂_μ) with the Euler derivatives (E). For example: for Maxwell theory it is $\partial_\mu E^\mu$, for Yang-Mills—$\partial_\mu E^{\mu(a)}$, for the second order metric GR-$\partial_\mu E^{\mu\nu}$. Variation of the action with respect to the fundamental (basic) fields of a theory defines a tensorial character of a differential identity. For the first order EC action the Euler derivatives are $E^{\mu(\alpha)} = \frac{\delta L_{EC}}{\delta e_{\mu(\alpha)}}$ and $E^{\mu(\alpha\beta)} = \frac{\delta L_{EC}}{\delta \omega_{\mu(\alpha\beta)}}$ [3]. Thus, the basic (the most fundamental) differential identities can be constructed starting from $\partial_\mu E^{\mu(\alpha)}$ and $\partial_\mu E^{\mu(\alpha\beta)}$ that lead to the following identities $I^{(\alpha)} = \partial_\mu E^{\mu(\alpha)} + \cdots$ and $I^{(\alpha\beta)} = \partial_\mu E^{\mu(\alpha\beta)} + \cdots$ (see [3]), which give rise to the *translational and rotational invariances in the tangent space* [11]. The Hamiltonian method applied to singular systems (the Dirac procedure) always leads to first class constraints that allow the restoration of the *gauge* invariance. The Hamiltonian formulation of the EC action leads to the first class constraints with the PB algebra that describe internal translation and rotation. This is clear from the first steps of the Dirac procedure and the tensorial character of the primary first class constraints [1, 2]. The same result, translational and rotational invariances, also follows from the analysis of basic differential identities at the Lagrangian level [3].

Finally, we would like to note that in the Lagrangian approach all symmetries that can be found seem to be on the same footing and what is only important, according to the Noether theorem [19], is to find the minimum number of independent symmetries (or, which is the same, differential identities—combinations of Euler derivatives). Why does the Hamiltonian formulation give exactly this minimum number of independent symmetries? And why as seems to be the case, do they always correspond to the basic differential identities of the Lagrangian formulation? These questions need to be answered even after proof of closure is completed and invariance is restored from the complete set of first class constraints. So, contrary to the authors of [51], we cannot conclude our paper by the unquestionable statement: "the case is closed".

[9] The constraint structure of the Dirac [17, 43] and Pirani, Schild and Skinner (PSS) [16, 50] Hamiltonian formulations leads to diffeomorphism and they are connected by canonical transformations [44]. But the ADM Hamiltonian is not related by any canonical transformation to the Dirac Hamiltonian (so also to PSS) and its gauge symmetry is not diffeomorphism. By performing the inverse Legendre transformation (eliminating momenta), the Dirac and PSS formulations lead back to the Einstein GR and the same equations of motion. In the case of the ADM formulation the equations of motion are different and it is not evident that they are equivalent to Einstein's equations of GR, for example, in Numerical Relativity it is claimed that the type of equations is changed from strongly hyperbolic (for Einstein's) to weakly hyperbolic for ADM equations (see [17] and references therein).

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Appendix A: *ABC* Properties

Here we collect properties of the ABC functions that were introduced in considering the Hamiltonian formulation of N-bein gravity [1, 2]. They also turn out to be very useful in the Lagrangian formalism [3].

These functions are generated by consecutive variation of the N-bein density

$$\frac{\delta}{\delta e_{\nu(\beta)}}\left(ee^{\mu(\alpha)}\right)=e\left(e^{\mu(\alpha)}e^{\nu(\beta)}-e^{\mu(\beta)}e^{\nu(\alpha)}\right)=eA^{\mu(\alpha)\nu(\beta)}, \tag{A.1}$$

$$\frac{\delta}{\delta e_{\lambda(\gamma)}}\left(eA^{\mu(\alpha)\nu(\beta)}\right)=eB^{\lambda(\gamma)\mu(\alpha)\nu(\beta)}, \tag{A.2}$$

$$\frac{\delta}{\delta e_{\tau(\sigma)}}\left(eB^{\lambda(\gamma)\mu(\alpha)\nu(\beta)}\right)=eC^{\tau(\sigma)\lambda(\gamma)\mu(\alpha)\nu(\beta)}, \tag{A.3}$$

$$\frac{\delta}{\delta e_{\varepsilon(\rho)}}\left(eC^{\tau(\sigma)\lambda(\gamma)\mu(\alpha)\nu(\beta)}\right)=eD^{\varepsilon(\rho)\tau(\sigma)\lambda(\gamma)\mu(\alpha)\nu(\beta)}, \quad \ldots \tag{A.4}$$

The first important property of these density functions is their total antisymmetry: interchange of two indices of the same nature (internal or external), e.g.

$$A^{\nu(\beta)\mu(\alpha)}=-A^{\nu(\alpha)\mu(\beta)}=-A^{\mu(\beta)\nu(\alpha)} \tag{A.5}$$

with the same being valid for B, C, D, etc. In particular, the presence of two equal indices of the same "nature" (both internal or both external) makes the functions A, B, etc. equal zero.

The second important property is their expansion using an external index

$$B^{\tau(\rho)\mu(\alpha)\nu(\beta)}=e^{\tau(\rho)}A^{\mu(\alpha)\nu(\beta)}+e^{\tau(\alpha)}A^{\mu(\beta)\nu(\rho)}+e^{\tau(\beta)}A^{\mu(\rho)\nu(\alpha)}, \tag{A.6}$$

$$\begin{aligned}C^{\tau(\rho)\lambda(\sigma)\mu(\alpha)\nu(\beta)}&=e^{\tau(\rho)}B^{\lambda(\sigma)\mu(\alpha)\nu(\beta)}-e^{\tau(\sigma)}B^{\lambda(\alpha)\mu(\beta)\nu(\rho)}\\&\quad+e^{\tau(\alpha)}B^{\lambda(\beta)\mu(\rho)\nu(\sigma)}-e^{\tau(\beta)}B^{\lambda(\rho)\mu(\sigma)\nu(\alpha)}\end{aligned} \tag{A.7}$$

or an internal index

$$B^{\tau(\rho)\mu(\alpha)\nu(\beta)}=e^{\tau(\rho)}A^{\mu(\alpha)\nu(\beta)}+e^{\mu(\rho)}A^{\nu(\alpha)\tau(\beta)}+e^{\nu(\rho)}A^{\tau(\alpha)\mu(\beta)}, \tag{A.8}$$

$$\begin{aligned}C^{\tau(\rho)\lambda(\sigma)\mu(\alpha)\nu(\beta)}&=e^{\tau(\rho)}B^{\lambda(\sigma)\mu(\alpha)\nu(\beta)}-e^{\lambda(\rho)}B^{\mu(\sigma)\nu(\alpha)\tau(\beta)}\\&\quad+e^{\mu(\rho)}B^{\nu(\sigma)\tau(\alpha)\lambda(\beta)}-e^{\nu(\rho)}B^{\tau(\sigma)\lambda(\alpha)\mu(\beta)}.\end{aligned} \tag{A.9}$$

The third property involves their derivatives

$$\left(eA^{\nu(\beta)\mu(\alpha)}\right)_{,\sigma}=\frac{\delta}{\delta e_{\lambda(\gamma)}}\left(eA^{\nu(\beta)\mu(\alpha)}\right)e_{\lambda(\gamma),\sigma}=eB^{\lambda(\gamma)\nu(\beta)\mu(\alpha)}e_{\lambda(\gamma),\sigma}, \tag{A.10}$$

$$\left(eB^{\tau(\rho)\nu(\beta)\mu(\alpha)}\right)_{,\sigma}=\frac{\delta}{\delta e_{\lambda(\gamma)}}\left(eB^{\tau(\rho)\nu(\beta)\mu(\alpha)}\right)e_{\lambda(\gamma),\sigma}=eC^{\tau(\rho)\lambda(\gamma)\nu(\beta)\mu(\alpha)}e_{\tau(\rho),\sigma}. \tag{A.11}$$

We also use the contraction of $B^{\tau(\rho)\mu(\alpha)\nu(\beta)}$ (A.6) with a covariant $e_{\tau(\lambda)}$:

$$e_{\tau(\lambda)}B^{\tau(\rho)\mu(\alpha)\nu(\beta)}=\tilde{\delta}^{\rho}_{\lambda}A^{\mu(\alpha)\nu(\beta)}+\tilde{\delta}^{\alpha}_{\lambda}A^{\mu(\beta)\nu(\rho)}+\tilde{\delta}^{\beta}_{\lambda}A^{\mu(\rho)\nu(\alpha)}. \tag{A.12}$$

The above properties considerably simplify the calculations.

Appendix B: Solution of the Equation of Motion for $\hat{\Sigma}$

To eliminate the $\hat{\Sigma}$ field in the course of the Lagrangian or Hamiltonian reduction, we perform variation of (58) or (62) with respect to $\hat{\Sigma}$ and solve this equation for $\hat{\Sigma}$. The corresponding part of the Lagrangian (Hamiltonian), quadratic and linear in $\hat{\Sigma}$, after changing dummy indices and performing some contractions is

$$L\left(\hat{\Sigma}\right)=eg_{qp}\hat{\Sigma}_k{}^{(mq)}\hat{\Sigma}_m{}^{(kp)}+2e\hat{\Sigma}_m{}^{(pq)}\hat{D}^m{}_{pq} \tag{B.1}$$

where $g_{qp}=e_{q(\alpha)}e^{(\alpha)}_p$ and

$$\hat{D}^m{}_{pq}=e^{m(\beta)}e_{p(\beta),q}+e^{m(\beta)}e^{(\gamma)}_qN_{p(\gamma\beta)0n(\sigma)}\pi^{n(\sigma)}. \tag{B.2}$$

Note that there are no symmetries in this expression in pq indices (e.g., $\hat{D}^m{}_{pq}\neq-\hat{D}^m{}_{qp}$, $\hat{D}^m{}_{pq}\neq\hat{D}^m{}_{qp}$), which is clear from its explicit form. Of course, we can do further contraction in the second term of (B.2); but to find the solution it is not necessary as it can be expressed in terms of the whole $\hat{D}^m{}_{pq}$ and the separation of it into contributions with momenta and spatial derivatives of covariant N-bein is sufficient at this stage and keep the expressions in compact form.

Variations of a traceless antisymmetric field (see the fundamental PB (64)) is

$$\frac{\delta\hat{\Sigma}_k{}^{(mq)}}{\delta\hat{\Sigma}_x{}^{(yz)}}=\delta^x_k\hat{\Delta}^{(mq)}_{(yz)}-\frac{1}{D-2}\left[\delta^x_y\hat{\Delta}^{(mq)}_{(kz)}-\delta^x_z\hat{\Delta}^{(mq)}_{(ky)}\right]. \tag{B.3}$$

It is clear from (B.3) that this expression is antisymmetric in mq and yz and equals to zero if the traces of $\hat{\Sigma}_k{}^{(mq)}$ or $\hat{\Sigma}_x{}^{(yz)}$ are taken. Variation of (B.1) gives

$$\hat{\Sigma}_y{}^{(px)}g_{zp}-\hat{\Sigma}_z{}^{(px)}g_{yp}=\hat{D}^x{}_{(yz)} \tag{B.4}$$

where

$$\hat{D}^x{}_{(yz)}=\hat{D}^x{}_{yz}-\hat{D}^x{}_{zy}-\frac{1}{D-2}\left[\delta^x_y\left(\hat{D}^m{}_{mz}-\hat{D}^m{}_{zm}\right)-\delta^x_z\left(\hat{D}^m{}_{my}-\hat{D}^m{}_{ym}\right)\right], \tag{B.5}$$

which is manifestly antisymmetric and traceless as it should be after variation with respect to the field with such properties.

To solve (B.4) we use Einstein's permutation [52]. To do this we must have three indices of the same "nature", either all external or all internal, and in the same position, covariant or contravariant. We can achieve this by contracting (B.4) with g_{wx}

$$g_{wx}\hat{\Sigma}_y{}^{(px)}g_{zp}-g_{wx}\hat{\Sigma}_z{}^{(px)}g_{yp}=g_{wx}\hat{D}^x{}_{(yz)}. \tag{B.6}$$

Now we have combinations with three free external indices in covariant position and can use the permutation $(wyz) + (yzw) - (zwy)$ that gives us

$$2g_{yx}\hat{\Sigma}_z{}^{(px)}g_{wp} = g_{wx}\hat{D}^x{}_{(yz)} + g_{yx}\hat{D}^x{}_{(zw)} - g_{zx}\hat{D}^x{}_{(wy)}. \tag{B.7}$$

To find explicitly $\hat{\Sigma}_z{}^{(px)}$, we have to use the Dirac inverse $\gamma^{km} = g^{km} - \frac{g^{0m}g^{0k}}{g^{00}}$. (We repeat, it is not a new variable, but a short-hand notation for a particular combination of N-bein fields). After contracting (B.7) with $\gamma^{ky}\gamma^{mw}$ we obtain the solution

$$\hat{\Sigma}_z{}^{(mk)} = \frac{1}{2}\left(\gamma^{kb}\hat{D}^m{}_{(bz)} - \gamma^{mb}\hat{D}^k{}_{(bz)} - \gamma^{ky}\gamma^{mw}g_{zx}\hat{D}^x{}_{(wy)}\right). \tag{B.8}$$

Of course, solution for antisymmetric and traceless field is antisymmetric (RHS of (B.8) is manifestly antisymmetric) and traceless (contracting (B.8) with δ^z_m or δ^z_k).

At this stage, we can check the $D = 3$ limit. The solution for $\hat{\Sigma}_z{}^{(mk)}$ (B.8) was obtained for all dimensions $D > 2$ and it has to vanish when $D = 3$. It is not difficult to check, taking $\hat{\Sigma}_z{}^{(mk)}$ from (B.8) and using the exact expressions of $\hat{D}^m{}_{(bz)}$ (B.5) and $\hat{D}^m{}_{pq}$ (B.2), that such a limit is preserved

$$\lim_{D=3}\hat{\Sigma}_z{}^{(mk)}(\pi) = \lim_{D=3}\hat{\Sigma}_z{}^{(mk)}\left(e_{,s}\right) = 0. \tag{B.9}$$

Actually, to demonstrate (B.9), it is not necessary to substitute the explicit form of $\hat{D}^m{}_{(bz)}$. When $D = 3$, there are only two independent components of $\hat{\Sigma}_z{}^{(mk)}$ and they are $\hat{\Sigma}_1^{(12)} = \hat{\Sigma}_2^{(12)} = 0$ because $\hat{\Sigma}_z{}^{(mk)}$ is antisymmetric and traceless with only spatial indices.

Substitution of (B.8) back into (58) or (74) gives the reduced Lagrangian or Hamiltonian, respectively.

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